Monotone Comparative Statics

When is the Argmax increasing?

Definitions

Fix $X \subseteq \mathbb{R}$ and $T \subseteq \mathbb{R}$ and a function $f : X \times T \to \mathbb{R}$

$$\phi(t) \ := \ \arg\max_{x \in X} f(x,t)$$

Take $x_h, x_\ell \in X$ with $x_h > x_\ell$, $t_h, t_\ell \in T$, $t_h > t_\ell$

Increasing Differences (ID)

$$f(x_h,t_h) - f(x_\ell,t_h) \geq f(x_h,t_\ell) - f(x_\ell,t_\ell)$$

Strict Increasing Differences (SID)

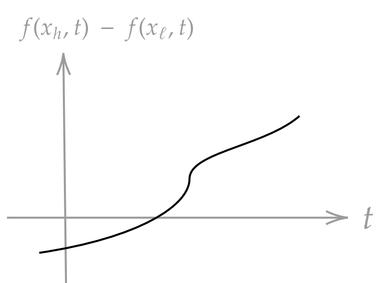
$$f(x_h,t_h) - f(x_\ell,t_h) > f(x_h,t_\ell) - f(x_\ell,t_\ell)$$

Single Crossing Property (SCP)

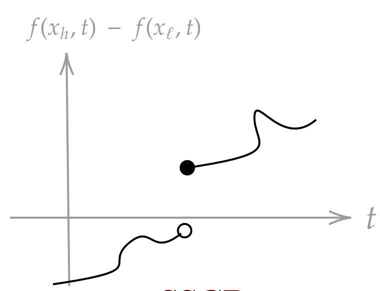
$f(x_h,t_\ell) - f(x_\ell,t_\ell) \geq 0$ implies $f(x_h,t_h) - f(x_\ell,t_h) \geq 0$
$f(x_h,t_\ell) - f(x_\ell,t_\ell) > 0$ implies $f(x_h,t_h) - f(x_\ell,t_h) > 0$

Strict Single Crossing Property (SSCP)

$f(x_h,t_\ell) - f(x_\ell,t_\ell) \geq 0$ implies $f(x_h,t_h) - f(x_\ell,t_h) > 0$



SID



SSCP

Facts

(1) Asuume f is differentiable then crospartials of f are strictly positive if and only if f has **SID**

(2) If f has **(S)ID** then f satisfies **(S)SCP**

(3) If f is constant in x or constant in t, then f has **ID**

(4) If f satisfies the **SSCP** and $g : \mathbb{R} \to \mathbb{R}$ srt. increasing then $g \circ f$ satisfies **SSCP**

(5) If f has **SID** and $g : \mathbb{R} \to \mathbb{R}$ strictly increasing does not imply $g \circ f$ has **SID**

(6) If f_1 has **SID** and f_2 has **ID** then $f_1 + f_2$ has **SID**

(7) If f_1 satisfies **(S)SCP** and f_2 satisfies **(S)SCP** does not imply $f_1 + f_2$ satisfies **(S)SCP***

*[3] shows that **(S)SCP** under addition may be preserved under certain conditions

Theorem 1

If f satisfies **SSCP**, $t_\ell < t_h$ $x_\ell \in \phi(t_\ell)$ and $x_h \in \phi(t_h)$ then $x_\ell \leq x_h$

Proof

Note that $f(x_\ell,t_\ell) - f(x_h,t_\ell) \geq 0$. So, if $x_\ell > x_h$ then, by **SSCP**, $f(x_\ell,t_h) - f(x_h,t_h) > 0$,which contradicts $x_h \in \phi(t_h)$. Thus, $x_\ell \leq x_h$.

Theorem 2

If f satisfies **SCP**, $t_\ell < t_h$, $x_\ell \in \phi(t_\ell)$, and $x_h \in \phi(t_h)$ then $\min(x_\ell, x_h) \in \phi(t_\ell)$ and $\max(x_\ell, x_h) \in \phi(t_h)$

Proof

If $x_h < x_\ell$ and $x_h \notin \phi(t_\ell)$ then $f(x_\ell,t_\ell) - f(x_h,t_\ell) > 0$. So, by **SCP**, $f(x_\ell,t_h) - f(x_h,t_h) > 0$, which contradicts $x_h \in \phi(t_h)$. Thus, either $x_\ell \leq x_h$ or $x_h \in \phi(t_\ell)$. In any case, $\min(x_\ell, x_h) \in \phi(t_\ell)$. The other conclusion is analogous.

Example

Alice's tree cutting problem

Alice owns a tree and must decide the day $x \in \mathbb{N}$ to cut it down and sell its wood. The size of the three at day x is given by a positive but not necessary monotone function $g : \mathbb{N} \to \mathbb{R}^+$. Alice is implatient with discount factor $\delta \in (0,1)$. So, her objective is to find the date $x \geq 0$ that maximizes

$$f(x,\delta) = \delta^x \cdot g(x).$$

Suppose that Alice suddenly becomes more patient and her discount factor changes from δ_0 to δ_1 with $\delta_1 > \delta_0$.
What can we say about solutions $x_0 \in \phi(\delta_0)$ and $x_1 \in \phi(\delta_1)$?

Solution: Note that the function $x\log(\delta)$ has **SID by (1)** and $\log(g(x))$ has **ID** by **(3)**. So, their sum satisfies **SID** by **(6)**. Thus, by **(3),** it also satisfies **SSCP. S**ince the $\exp$ is increasing, **(4)** implies

$$\begin{aligned} f(x,\delta) &= \delta^x \cdot \log(g(x)) \\ &= \exp(x\log(\delta) + \log(g(x))) \end{aligned}$$

aslo satisfies **SSCP.**

Since $\delta_0 < \delta_1$, it follows by Theorem 1 that $x_0 \leq x_1$. So, the more patient Alice becomes, the longer she waits to cutdown the tree.

References

[1] Van Zandt, Timothy. "An introduction to monotone comparative statics." Insead, Fontainebleau (2002).

[2] Milgrom, Paul, and Chris Shannon. "Monotone comparative statics." Econometrica: Journal of the Econometric Society (1994): 157-180.

[3] Quah, John K-H., and Bruno Strulovici. "Aggregating the single crossing property." Econometrica 80.5 (2012).

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