Blackwell's Theorem

When is one source of information more informative than another?

Decision problem

States: Θ (finite). **Decision problem** (for Θ), denoted $\mathrm{DP} = (A, u, p)$:

- compact action set A;
- continuous utility function $u : A \times \Theta \to \mathbb{R}$;
- prior $p \in \Delta(\Theta)$.

Objective:

$$\max_{a \in A} \sum_{\theta} p(\theta) \cdot u(a, \theta)$$

Experiments

Experiment: $E = (S, \pi)$, with a finite set of signals S and a conditional distribution $\pi : \Theta \to \Delta(S)$ (a Markov matrix, rows θ).

Strategy for E: a mapping $\alpha : S \to \Delta(A)$; Σ_E is the set of strategies for E.

Expected utility for DP given E:

$$V(E, \mathrm{DP}) = \max_{\alpha \in \Sigma_E} \sum_{\theta, a, s} u(a, \theta) \cdot \alpha(a \mid s) \cdot \pi(s \mid \theta) \cdot p(\theta)$$

Garbling & more information

Definition. An experiment $E' = (S', \pi')$ is a **garbling** of $E = (S, \pi)$ if there is a mapping $\gamma : S \to \Delta(S')$ such that

$$\pi'(s' \mid \theta) = \sum_{s \in S} \gamma(s' \mid s) \cdot \pi(s \mid \theta)$$

E' "adds" noise to E.

Definition. E is **more informative** than E' if for any decision problem DP,

$$V(E, \mathrm{DP}) \;\geq\; V(E', \mathrm{DP})$$

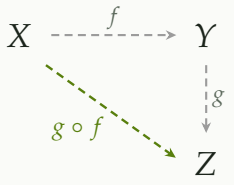
Any Bayesian agent prefers to observe E rather than E'.

Both define a partial order on the set of experiments: not all pairs of experiments are ranked.

Category of stochastic mappings

Objects: the finite sets Θ, A, S, S'. A morphism $f : X \dashrightarrow Y$ is a function $f : X \to \Delta(Y)$.

- **Composition:** $g \circ f\,(z \mid x) = \sum_y g(z \mid y)\, f(y \mid x)$;
- **Identity:** $\mathrm{Id}_X(x \mid x') = 1$ iff $x = x'$.



Strategies and experiments live in the same category, so E' is a garbling of E iff some $\gamma : S \dashrightarrow S'$ satisfies $\gamma \circ \pi = \pi'$. Set

$$\Lambda(E, A) := \{\alpha \circ \pi \mid \alpha : S \dashrightarrow A\}$$

Posterior beliefs

An element $F \in \Delta(\Delta(\Theta))$ is a **distribution of beliefs**. For a prior p, an experiment induces one: the law of the posterior $Z_\theta := \mathbb{E}[\mathbf{1}_\theta[\Theta] \mid S]$.

Definition. F is a **mean-preserving spread** (MPS) of $\widetilde{F}$ if there is a probability space with $\Delta(\Theta)$-valued random variables $Z, \widetilde{Z}$ such that $Z \sim F$, $\widetilde{Z} \sim \widetilde{F}$ and $\mathbb{E}(Z \mid \widetilde{Z}) = \widetilde{Z}$.

Definition. F dominates $\widetilde{F}$ in the **convex order** if for every continuous convex $h : \Delta(\Theta) \to \mathbb{R}$,

$$\int_{\Delta(\Theta)} h(p)\, dF(p) \;\geq\; \int_{\Delta(\Theta)} h(p)\, d\widetilde{F}(p).$$

Two states

Conditions (1)–(5) are hard to verify when $|\Theta| \geq 3$, but (4) is easy when $\Theta = \{0, 1\}$. Let $F, \widetilde{F}$ be the cdfs of the posteriors $\mathbb{P}(\Theta = 1 \mid S)$, $\mathbb{P}(\Theta = 1 \mid \widetilde{S})$ on $[0, 1]$.

Lemma. F is a mean-preserving spread of $\widetilde{F}$ iff $\mathbb{E}_F[Z] = \mathbb{E}_{\widetilde{F}}[Z]$ and

$$\int_0^z F(t)\, dt \;\geq\; \int_0^z \widetilde{F}(t)\, dt \quad \text{for each } z \in [0, 1].$$

Theorem (Blackwell)

The following rankings are equivalent:

(1) INFORMATION E' is a garbling of E.

(2) POSSIBILITIES For any action set A, $\Lambda(E', A) \subseteq \Lambda(E, A)$.

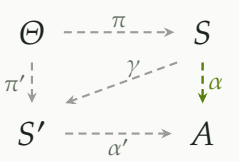
(3) PAYOFFS E is more informative than E': $V(E, \mathrm{DP}) \geq V(E', \mathrm{DP})$ for each DP.

(4) POSTERIOR BELIEFS For any prior $p \in \Delta(\Theta)$, if F and F' are the distributions of posterior beliefs induced by E and E', then F is a **MPS** of F'.

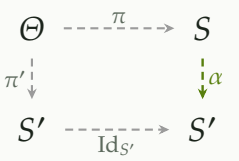
(5) CONVEX OPTIMIZATION For any prior $p \in \Delta(\Theta)$, if F and F' are as in (4), then F dominates F' in the **convex order**.

Proof: (1) ⇔ (2) ⇔ (3)

(1)⇒(2) Given $\alpha' : S' \dashrightarrow A$, set $\alpha = \alpha' \circ \gamma$; then $\alpha \circ \pi = \alpha' \circ \gamma \circ \pi = \alpha' \circ \pi'$, so the diagram commutes.



(2)⇒(1) Take $A = S'$ and $\alpha' = \mathrm{Id}_{S'}$. By (2) some α makes the diagram commute: $\pi' = \pi' \circ \mathrm{Id}_{S'} = \alpha \circ \pi$.

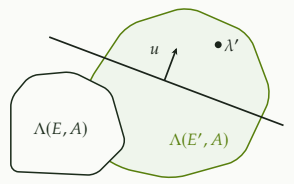


(2)⇒(3) Since $V(E, \mathrm{DP}) = \max_{\lambda \in \Lambda(E,A)} \sum_{\theta,a} u(a, \theta)\, \lambda(a \mid \theta)\, p(\theta)$, the inclusion $\Lambda(E', A) \subseteq \Lambda(E, A)$ gives $V(E, \mathrm{DP}) \geq V(E', \mathrm{DP})$.

(3)⇒(2) Suppose some $\lambda' \in \Lambda(E', A) \setminus \Lambda(E, A)$ exists. As $\Lambda(E, A) \subset \mathbb{R}^{\Theta \times A}$ is compact and convex, there is a hyperplane $u \in \mathbb{R}^{\Theta \times A}$ such that, for each $\lambda \in \Lambda(E, A)$,

$$\sum_{\theta,a} u(\theta, a)\, \lambda(a \mid \theta) \;<\; \sum_{\theta,a} u(\theta, a)\, \lambda'(a \mid \theta).$$

Hence, for $\mathrm{DP} = (A, u, p)$ with **uniform** p, we have $V(E, \mathrm{DP}) < V(E', \mathrm{DP})$, contradicting (3).



Proof: (1) ⇒ (4) ⇒ (5) ⇒ (3)

(1)⇒(4) Fix p; on $\Omega = \Theta \times S \times S'$ let $\mathbb{P}[\theta, s, s'] = p(\theta)\, \pi(s \mid \theta)\, \gamma(s' \mid s)$ and $Z_\theta = \mathbb{E}[\mathbf{1}_\theta[\Theta] \mid S]$, $Z'_\theta = \mathbb{E}[\mathbf{1}_\theta[\Theta] \mid S']$. As $\Theta \dashrightarrow S \dashrightarrow S'$, the states Θ and S' are conditionally independent given S, so $Z_\theta = \mathbb{E}[\mathbf{1}_\theta[\Theta] \mid S, S']$. By the law of iterated expectations

$$\mathbb{E}[Z_\theta \mid S'] = \mathbb{E}[\mathbf{1}_\theta[\Theta] \mid S'] = Z'_\theta,$$
$$\mathbb{E}[Z_\theta \mid Z'] = \mathbb{E}\big[\mathbb{E}[Z_\theta \mid S'] \,\big|\, Z'\big] = Z'_\theta.$$

(4)⇒(5) If $\mathbb{E}[Z \mid Z'] = Z'$ and h is convex and continuous, Jensen gives

$$\int h\, dF = \mathbb{E}[h(Z)] = \mathbb{E}\big[\mathbb{E}[h(Z) \mid Z']\big]$$
$$\geq \mathbb{E}\big[h(\mathbb{E}[Z \mid Z'])\big] = \mathbb{E}[h(Z')] = \int h\, dF'.$$

(5)⇒(3) For $\mathrm{DP} = (A, u, p)$ let $h(q) = \max_{a \in A} \sum_\theta q(\theta) u(a, \theta)$, a pointwise maximum of linear functions, hence continuous and convex. Then $V(E, \mathrm{DP}) = \int h\, dF \geq \int h\, dF' = V(E', \mathrm{DP})$.

Application: a seller's revenue

A buyer's value $v \sim U[0, 1]$ is unobserved. The seller observes an experiment of v, posts a take-it-or-leave-it price p, and the buyer buys iff $v \geq p$. **Claim:** expected revenue lies in $[\frac{1}{4}, \frac{1}{2}]$.

- **Best** (fully revealing): $p = v$, revenue $\mathbb{E}[v] = \frac{1}{2}$.
- **Worst** (no information): $p = \frac{1}{2}$, sale probability $\frac{1}{2}$, revenue $\frac{1}{4}$.

Any experiment is a garbling of the fully revealing one, and observing nothing is a garbling of any experiment: by Blackwell the revenue is bounded by these two.

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