Notes on Sequential Games and Perfect Recall*

Amanda Friedenberg† Ernesto Rivera Mora‡

March 22, 2026

1 Sequential Games

Mathematical Preliminaries Fix a non-empty set X and let $V \subseteq \bigcup_{n\in\mathbb{N}} X^n \cup X^{\mathbb{N}}$ be a set of finite or infinite sequences on X, including possibly the empty sequence $\emptyset$. Define a binary precedence relation $\preceq$ on V so that $v \preceq w$ if and only if v is an initial segment of w; this allows $v = w$. If $v \preceq w$ but $v \neq w$, write $v \prec w$. The empty sequence $\emptyset$ is taken to be an initial segment of each $v \in V$.

Call $(V, \preceq)$ a **tree** (**on X**) if $w \in V$ and $v \preceq w$ implies $v \in V$. In this case, $v \in V$ is called a *node*. The *immediate predecessor* of v is the unique node $w \in V$ so that $v = (w, x)$ for some $x \in X$. A node $w \in V$ is an *immediate successor* of $v \in V$ if v is an immediate predecessor of w. Write $\mathrm{succ}(v)$ for the set of immediate successors of v. Write $\mathrm{term}(V)$ for the set of *terminal nodes* in V, i.e., the set of $v \in V$ so that $\mathrm{succ}(v) = \emptyset$.

Given a set of indices I and sets X_i, write $X = \prod_{i\in I} X_i$ and $X_{-i} = \prod_{j\in I\setminus\{i\}} X_j$.

Extensive-Form Game This definition of an extensive-form game is adapted from Osborne and Rubinstein (1994); the adaptation allows for both simultaneous moves (Section 6.3.2) and imperfect information (Section 11.1). It also allows for imperfect information at the end of the game, which is important for psychological games.

An **extensive-form game** is given by $\Gamma = (I, A_i, (V, \preceq), H_i, \Pi_i : i \in I)$, described below. First, $I = \{1, \ldots, |I|\}$ is a finite set of players. Second, each A_i is a non-empty set of *actions* for player i. Third, $(V, \preceq)$ is a *tree* on $A = \prod_{i\in I} A_i$ that satisfies the following criterion: If $(v, a_i, a_{-i}), (v, b_i, b_{-i}) \in V$, then $(v, b_i, a_{-i}) \in V$. That is, the actions that player i can choose at v do not depend on the actions other players can choose at v. Write $Z = \mathrm{term}(V)$ for the set of terminal nodes of $(V, \preceq)$.

Fourth, each H_i is a partition of V; a member of H_i, written h_i, is an *information set* for i. The information partition satisfies two criteria. First, if $v, w \in h_i$ and $(v, a_i, a_{-i}) \in V$, then there exists some $b_{-i} \in A_{-i}$ so that $(w, a_i, b_{-i}) \in V$. That is, if a_i is an action available to i at a node in h_i, then a_i is an action available at every node in h_i. Second, if $Z \cap h_i \neq \emptyset$, then $h_i \subseteq Z$. That is, the game has an *observable end*.

*We thank Frank Cheng for his feedback. These notes are pedagogical in nature and are not intended as research output.

†University of Michigan, amanda@amandafriedenberg.org

‡University of Colorado, Boulder ernesto.riveramora@colorado.edu

Finally, each $\Pi_i : Z \to \mathbb{R}$ is an *extensive-form payoff function* for i. The extensive-form payoff function of i is measurable with respect to H_i, i.e., if $z, z' \in h_i$ then $\Pi_i(z) = \Pi_i(z')$.

Background Idea This definition generalizes that in Osborne and Rubinstein (1994) to provide a single definition that allows for simultaneous-moves and imperfect information. As in Osborne and Rubinstein, each node corresponds to a sequence of actions played to reach the node. If a player does not move at a given non-terminal node, the player is modeled as having a "dummy move" at that node. That is, if i does not move at a node $v \in V \backslash Z$, the set of actions available to i at v, viz. $A_i(v)$, is a singleton. Player i is *inactive* (resp. *active*) at $v \in V \backslash Z$ if $|A_i(v)| = 1$ (resp. if $|A_i(v)| \geq 2$). Note, the set of actions available at a terminal node $v \in Z$ is $A_i(v) = \emptyset$; that is, terminal nodes are neither active nor inactive.

While each player moves at each non-terminal node v, they may have different information at any given node. In particular, we can have information sets of player 1, h_1, and player 2, h_2, with $v \in h_1 \cap h_2$ even though $h_1 \neq h_2$.

If $v, w \in h_i$, then $A_i(v) = A_i(w)$ and so we can write $A_i(h_i)$ for the set of actions available at h_i. Say i is *inactive* (resp. *active*) at $h_i \in H_i$ if $|A_i(h_i)| = 1$ (resp. if $|A_i(h_i)| \geq 2$). An important conceptual point is in order, which will guide definitions to come: If i is inactive at h_i, then i does not move at the nodes in h_i nor does i receive information at h_i. Instead, i's choice is simply a modeling device that allows us to view nodes as a sequence of actions in A^n. Under this interpretation, it is most natural to focus on *non-trivial games*, i.e., games where at each player is active at each of their non-terminal information sets. However, the definitions/results do not require non-triviality.

In Section 3, we discuss how to modify the definition of an extensive-form to allow for two types of inactive moves: Those that are simply a modeling device and those at which the inactive player receives information.

Examples We next give examples that illustrate the framework.

Example 1.1 (Repeated Game)**.** Fix a simultaneous-move game, $(A_i, u_i : i \in I)$, where A_i is the set of actions of player i and $u_i : A \to \mathbb{R}$ is the payoff function of i. This simultaneous move game induces a T-period repeated game with perfect monitoring. In that game, the set nodes is

$$V = \begin{cases} \bigcup_{t=0}^{T} A^t & \text{if } T \text{ is finite} \\ \bigcup_{t \geq 0} A^t \cup A^{\mathbb{N}} & \text{if } T \text{ is infinite.} \end{cases}$$

Terminal nodes are elements of A^T when T is finite and sequences in $A^{\mathbb{N}}$ when T is infinite. Each player i is active at each node $v \in V \backslash Z$. The set of information sets for i is

$$H_i = \big\{\{v\} : v \in V\big\}.$$

The payoff functions $\Pi_i : Z \to \mathbb{R}$ are derived from u_i in the usual way.

A repeated game of perfect monitoring is a specific example where there are observable actions. In any extensive-form game with observable actions all players' information sets are singletons and each player has the same set of information sets. A game of perfect information is a game with observable actions so that, at each node, there is at most one active player. The next example illustrates how we model games of perfect information.

Example 1.2. Consider a two-player extensive-form game. Player 1 moves first, choosing L or R. If Player 1 chooses L, Player 2 selects ℓ or r. If Player 1 chooses R, the game ends.

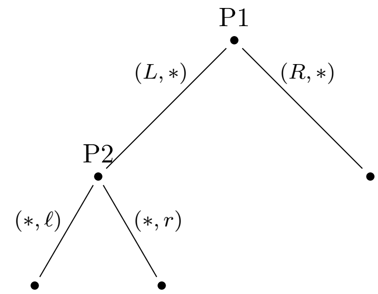

Figure 1.1 Perfect Information Game

Figure 1.1 depicts this pictorally. At both non-terminal nodes, both players move. However, at the root, Player 1 is the only active player. (Thus, the root is labeled P1.) Player 2 must choose the dummy action $*$, indicating that Player 2 does not really have a choice. At the node $(L, *)$, Player 2 is the only active player. (Thus, that node is labeled P2.) Player 1 must choose the dummy action $*$, indicating that Player 1 does not have a choice.

The next example illustrates a game of imperfect information.

Example 1.3. Consider the game in Figure 1.2. At each node, there is a single player that is active. P1 moves first and assigns the play to either P2 or P3. The selected player then chooses an action: either passing the play to the other player or ending the game. The labels at each node indicate which agent is active at that node. All information sets of P1 are singletons. P2 has a single non-singleton information set, consisting of the two nodes where P2 is active. Similarly, P3 has a single non-singleton information set containing the two nodes where P3 is active. As a result, P2 and P3 cannot distinguish whether they are moving first or second.

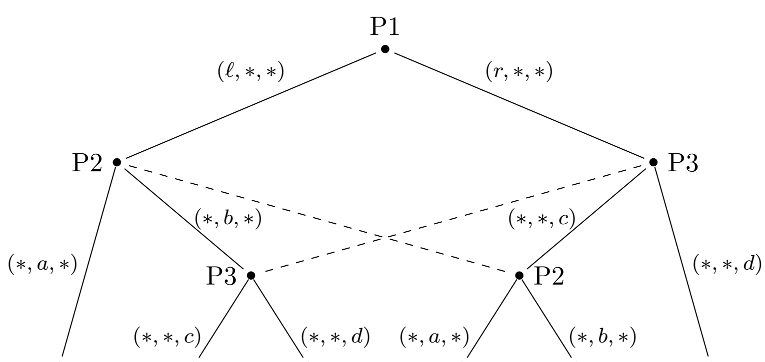

Figure 1.2 Imperfect Information Game

Induced Strategic Form A (pure) **strategy** for player i is a mapping $s_i : H_i \to A_i$ so that $s_i(h_i) \in A_i(h_i)$ for each $h_i \in H_i$. Write S_i for the set of strategies for player i. There is a mapping $\zeta : S \to Z$ so that $\zeta(s)$ specifies the terminal node reached by the strategy profile $s \in S$. The extensive-form game Γ induces a strategic form $G = (S_i, \pi_i : i \in I)$, where $\pi_i = \Pi_i \circ \zeta$ for each $i \in I$.

2 Recall

There are two recall properties that are typically imposed on the extensive-form game: no absent-mindedness and perfect recall.

Definition 2.1. Say Γ satisfies **no absentmindedness** if the following holds: If $v \prec w$ and i is active at v, then there is no $h \in H_i$ with $\{v, w\} \subseteq h$.

No absentmindedness says that, if two nodes v, w are in the same information set of player i and i is active at v, then v does not strictly precede w. To understand the condition, recall, if $\{v, w\} \subseteq h$ and i is active at v, then i is active at w. If it is also the case that $v \prec w$, when i moves at w, i cannot remember if she previously moved at v or not.

Definition 2.2. Say Γ satisfies **perfect recall** if, for any pair of distinct information sets $h, h' \in H_i$, the following property is satisfied: If $\{v, w\} \subseteq h$ and $v' \in h'$ with $(v', a_i, a_{-i}) \preceq v$, then either i is inactive at v' or there exists a node $w' \in h'$ so that $(w', a_i, a'_{-i}) \preceq w$ for some $a'_{-i} \in A_{-i}$.

Remark 2.1. The typical definition of perfect recall would require that "there exists a node $w' \in h'$ so that $(w', a_i, a'_{-i}) \preceq w$ for some $a'_{-i} \in A_{-i}$." Here we only impose the requirement if i is active.

To understand why, return to Example 1.3. Consider the nodes

$$v = \big((\ell, *, *), (*, b, *)\big) \qquad \text{and} \qquad w = (r, *, *).$$

Both v and w are in the same information set for P3, h_3. It is also the case that $v' = (\ell, *, *)$ precedes v and there is an information set of P3 with $h'_3 = \{v'\}$. But there is no member of h'_3 that precedes w.

Nonetheless, the example does not fail perfect recall, either substantively or according to our definition. In particular, while technically P3 can make a dummy choice at v', pragmatically P3 does not move at v' nor does P3 get information at v'. That is, P3 is inactive at v'. Our definition of perfect recall is automatically met because P3 is inactive.

Implications of Perfect Recall We now point out that standard implications of perfect recall hold with these specific definitions. Proposition 2.1 is based on insights in Brandenburger (2007). Propositions 2.2-2.3 are based on standard results. (We do not know the original source.)

Write $S(v)$ for the set of strategy profiles that passes through a node $v \in V$ and write $S_i(v) = \mathrm{proj}_{S_i} S(v)$. Similarly, fix $h \in \bigcup_{i \in I} H_i$ and write $S(h)$ for the set of strategy profiles that passes through some node in h. The set $S(h)$ is then the set of strategy profiles that reach h. Write $S_i(h) = \mathrm{proj}_{S_i} S(h)$ and $S_{-i}(h) = \mathrm{proj}_{S_{-i}} S(h)$. So, $S_i(h)$ is the set of strategies of i that allow h and $S_{-i}(h)$ is the set of strategy profiles of players other than i which allow h.

Definition 2.3 (Kuhn (1953)). Say Γ satisfies **Kuhn perfect recall** if, for each h_i, the following property is satisfied: If $s_i \in S_i(v)$ for some $v \in h_i$, then $s_i \in S_i(w)$ for all $w \in h_i$.

Proposition 2.1. *A game Γ satisfies no absentmindedness and perfect recall if and only if it satisfies Kuhn perfect recall.*

Proof. Suppose, first, that Γ satisfies no absentmindedness and perfect recall. Fix an information set $h \in H_i$ and $v, w \in h$. Suppose $(s_i, s_{-i}) \in S(v)$; we will show that $s_i \in S_i(w)$.

To do so, fix some $(r_i, r_{-i}) \in S(w)$. If $\zeta(s_i, r_{-i}) = \zeta(r_i, r_{-i})$, then certainly $(s_i, r_{-i}) \in S(w)$. So suppose $\zeta(s_i, r_{-i}) \neq \zeta(r_i, r_{-i})$. Let $\hat{w}$ be the last common predecessor of $\zeta(s_i, r_{-i})$ and $\zeta(r_i, r_{-i})$. Note, since $(r_i, r_{-i}) \in S(\hat{w}) \cap S(w)$, it follows that either $w \preceq \hat{w}$ or $\hat{w} \prec w$. It suffices to show that $w \preceq \hat{w}$: If so, $(s_i, r_{-i}) \in S(\hat{w}) \subseteq S(w)$, establishing $s_i \in S_i(w)$.

Suppose, contra hypothesis, that $\hat{w} \prec w$. Let $\hat{w} \in \hat{h}$, where $\hat{h} \in H_i$. Since $\hat{w}$ is the last common predecessor of $\zeta(s_i, r_{-i})$ and $\zeta(r_i, r_{-i})$, it follows that $s_i(\hat{h}) \neq r_i(\hat{h})$. Thus, i is active at $\hat{h}$. So, by no absentmindedness and the fact that $\hat{w} \prec w$, $\hat{h} \neq h$. As such, by perfect recall, there exists some $\hat{v} \in \hat{h}$ so that $(\hat{v}, r_i(\hat{h}), a_{-i}) \preceq v$ for some a_{-i}. Since $(s_i, s_{-i}) \in S(v)$, it follows that $(s_i, s_{-i}) \in S(\hat{v}, r_i(\hat{h}), a_{-i})$. But this contradicts $s_i(\hat{h}) \neq r_i(\hat{h})$.

Next, suppose that Γ fails no absentmindedness. That is, there exists an information set $h \in H_i$ at which i is active and nodes $\{v, w\} \subseteq h$ so that $v \prec w$. Choose v so that there is no $v' \in h$ with $v' \prec v$. Since i is active at h, there exists an action $a_i \in A_i(v)$ so that there is no a_{-i} with $(v, a_i, a_{-i}) \preceq w$. Because there is no $v' \in h$ such that $v' \prec v$, we can find a strategy profile $s_i \in S_i(v)$ with $s_i(h) = a_i$. Since $s_i \notin S_i(w)$, Kuhn perfect recall fails.

Finally, suppose Γ fails perfect recall. We will argue that it also fails Kuhn perfect recall. Since Γ fails perfect recall, we can find distinct information sets $h, h' \in H_i$ where i is active at h' and the following holds: There are nodes $v, w \in h$ and a node $v' \in h'$ with some $(v', a_i, a_{-i}) \preceq v$. But there is no node $w' \in h'$ with $(w', a_i, b_{-i}) \preceq w$ for some $b_{-i} \in A_{-i}$.

First suppose that there exists some $w' \in h'$ so that $w' \preceq w$. Since $h \neq h'$, $w' \neq w$ and so there exists $(b_i, b_{-i}) \in A$ with $(w', b_i, b_{-i}) \preceq w$. Given that Γ fails perfect recall, $b_i \neq a_i$. So, if $s_i \in S_i(w)$ then $s_i \notin S_i(v)$. This contradicts Kuhn perfect recall.

For the remainder, suppose there is no $w' \in h'$ so that $w' \preceq w$. Let $\hat{w}$ be the last common predecessor of v' and w. By definition, $\hat{w} \preceq v'$ and $\hat{w} \preceq w$. Since there is no $w' \in h'$ with $w' \preceq w$, $\hat{w} \neq v'$ and so $\hat{w} \prec v'$. There are two cases to consider:

Case 1: $\hat{w} \neq w$. Then $\hat{w} \prec w$. There exist action profiles so that $(\hat{w}, \hat{a}_i, \hat{a}_{-i}) \preceq v'$ and $(\hat{w}, \hat{b}_i, \hat{b}_{-i}) \preceq w$. If $\hat{a}_i \neq \hat{b}_i$ and $s_i \in S_i(w)$ then $s_i \notin S_i(v')$. Since $S(v) \subseteq S(v')$, $s_i \notin S_i(v)$ contradicting Kuhn perfect recall. If $\hat{a}_i = \hat{b}_i$, we can find an $s_i \in S_i(w) \cap S_i(v')$ so that $s_i(h') \neq a_i$. As such $s_i \in S_i(w)$ but $s_i \notin S_i(v)$, again contradicting Kuhn perfect recall.

Case 2: $\hat{w} = w$. Then $w \prec v'$ and so $S_i(v') \subseteq S_i(w)$. But, since i is active at v', there is a strategy $s_i \in S_i(v')$ where $s_i(h') \neq a_i$. Then $s_i \in S_i(w)$, while $s_i \notin S_i(v)$, contradicting Kuhn perfect recall. $\square$

Proposition 2.2. *Fix a game Γ that satisfies perfect recall. If i is active at $h, h' \in H_i$, then either $S(h) \subseteq S(h')$, $S(h') \subseteq S(h)$, or $S(h) \cap S(h') = \emptyset$.*

Example 2.1. The following shows why it is important that i is active at both $h, h' \in H_i$. Consider the extensive-form game in the Figure 2.1. P1 is active at $\emptyset$ and u, while P2 is active at v and w. The information sets of P1 are all singletons, while the set of information sets for P2 are

$$H_2 = \{\{\emptyset\}, \{u\}, \{w, v\}, \{z_1\}, \{z_2\}, \{z_3\}, \{z_4\}\}.$$

Thus, both $\{u\}$ and $h_2 = \{v, w\}$ are information sets of P2.

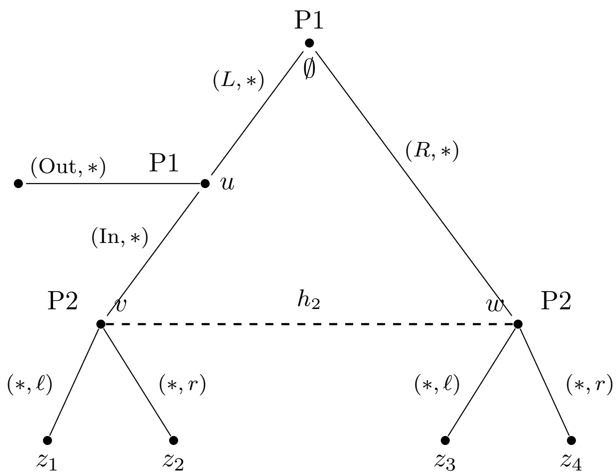

Figure 2.1 Illustration of Proposition 2.2

Notice,

$$S(\{u\}) = \big\{(L-\mathrm{Out}-*, *-*-\ell), (L-\mathrm{Out}-*, *-*-r), (L-\mathrm{In}-*, *-*-\ell), (L-\mathrm{In}-*, *-*-r)\big\}$$

while

$$S(h_2) = S \backslash \big\{(L-\mathrm{Out}-*, *-*-\ell), (L-\mathrm{Out}-*, *-*-r)\big\}.$$

So $S(\{u\}) \cap S(h_2) \neq \emptyset$, even though they are not nested.

Proof of Proposition 2.2. Throughout fix $h, h' \in H_i$ where $h \neq h'$ and i is active at both h and h'. First suppose that there exists $v \in h$ and $v' \in h'$ so that $v' \prec v$. That is, there exists (a_i, a_{-i}) with $(v', a_i, a_{-i}) \preceq v$. We will show that $S(h) \subseteq S(h')$. To see this, fix some $s \in S(h)$ and note that there exists $w \in h$ so that $s \in S(w)$. Since i is active at h', by perfect recall, there exists some $w' \in h'$ and some (a_i, b_{-i}) so that $(w', a_i, b_{-i}) \preceq w$. Since $s \in S(w) \subseteq S(w')$ and $w' \in h'$, $s \in S(h')$.

An analogous argument shows that, if $v \in h$ and $v' \in h'$ so that $v \prec v'$, then $S(h') \subseteq S(h)$. So, suppose that, for all $v \in h$ and $v' \in h'$, neither $v \prec v'$ nor $v' \prec v$. Fix some $v \in h$ and $v' \in h'$; note, v and v' must be distinct nodes since $h \neq h'$. Let $\hat{v}$ be the last common predecessor of v and v'. It follows that there is some player j and some $(\hat{v}, a_j, a_{-j}) \preceq v$, $(\hat{v}, a'_j, a'_{-j}) \preceq v'$ with $a_j \neq a'_j$. Let $\hat{h}_j \in H_j$ be the information set of j with $\hat{v} \in \hat{h}_j$ and note: If $(s_j, s_{-j}) \in S(v)$ and $(s'_j, s'_{-j}) \in S(v')$

then $s_j(\hat{h}_j) \neq s'_j(\hat{h}_j)$. It follows that $S(v) \cap S(v') = \emptyset$. Since v and v' were chosen arbitrarily, it follows that $S(h) \cap S(h') = \emptyset$. $\square$

Proposition 2.3. *Fix a game Γ that satisfies no absentmindedness and perfect recall. If $h \in H_i$, then $S(h) = S_i(h) \times S_{-i}(h)$.*

Proof. Fix $h \in H_i$ and note $S(h) \subseteq S_i(h) \times S_{-i}(h)$. We will show that, if $s_i \in S_i(h)$ and $r_{-i} \in S_{-i}(h)$, then $(s_i, r_{-i}) \in S(h)$.

To do so, note that there exists $s_{-i} \in S_{-i}$ and $r_i \in S_i$ so that $(s_i, s_{-i}), (r_i, r_{-i}) \in S(h)$. This implies that there are nodes $v, w \in h$ with $(s_i, s_{-i}) \in S(v)$ and $(r_i, r_{-i}) \in S(w)$. If $v = w$ certainly $(s_i, r_{-i}) \in S(v) \subseteq S(h)$. So suppose that $v \neq w$ and let $\hat{v}$ be the last common predecessor of v and w. Note, $\hat{v} \preceq v$ and $\hat{v} \preceq w$.

First, suppose that $\hat{v} = w$. Since $w = \hat{v} \preceq v$, $S(v) \subseteq S(w)$. As such, $(s_i, s_{-i}), (r_i, r_{-i}) \in S(w)$ and so $(s_i, r_{-i}) \in S(w)$. Second, suppose that $\hat{v} = v$. Since $\hat{v} \preceq w$ and $v \neq w$, it follows that $v \prec w$. Since $v, w \in h \in H_i$, by no absentmindedness, i cannot be active at $\hat{v} = v$. As a consequence $s_i(h) = r_i(h)$. Since $(s_i, s_{-i}) \in S(v)$, $(r_i, r_{-i}) \in S(w) \subseteq S(v)$ and $s_i(h) = r_i(h)$, it must be the case that $(s_i, r_{-i}) \in S(w)$.

Finally, suppose that $\hat{v} \neq w, v$, i.e., $\hat{v} \prec v$ and $\hat{v} \prec w$. Consider any node w' with $\hat{v} \preceq w' \prec w$ and $w' \in h' \in H_i$. By perfect recall, there must be some $v' \in h'$ so that $(v', a'_i, a'_{-i}) \prec v$ if and only if $(w', a'_i, b'_{-i}) \prec w$ for some $b'_{-i} \in A_{-i}$. As a consequence of the fact that $s_i \in S_i(v)$, on the path from $\hat{v}$ to w, s_i only makes choices that allow w. It follows that $(s_i, r_{-i}) \in S(w)$. $\square$

3 An Alternative Approach

In the approach above, we took the position that any node at which player i is inactive represents a situation where player i does not receive information nor does i move. We can modify the approach to allow a player to both get information and make a trivial choice at *some* nodes at which they are inactive.

Now, an **extensive-form game** is given by $\Gamma^* = (I, A_i, (V, \preceq), N_i, H_i, \Pi_i : i \in I)$, where I, A_i, $(V, \preceq)$, and Π_i are defined as above. The key difference lies in a set of *informational nodes for each player i*, viz. $N_i \subseteq V$. This set has the property that (i) $Z \subseteq N_i$, and (ii) if $v \in V \backslash N_i$, then $|A_i(v)| \leq 1$. That is, if v is not an informational node for i, then v is a non-terminal node at which i is not active. Then, the set H_i is a partition of N_i (and not V). Note, the information partition satisfies the same criteria described above. In particular, if $v, w \in h_i$ and $(v, a_i, a_{-i}) \in V$, then there exists some $b_{-i} \in A_{-i}$ so that $(w, a_i, b_{-i}) \in V$; and if $Z \cap h_i \neq \emptyset$, then $h_i \subseteq Z$.

Under this approach, a node can be classified into three distinct types for each player i:

(i) *Nodes at which player i makes genuine choices:* These are nodes $v \in N_i$ where player i is active (i.e., $|A_i(v)| \geq 2$). At such nodes, i can (but need not) receive information.

(ii) *Nodes at which player i receives information but only has a trivial choice:* These are nodes $v \in N_i$ where player i is inactive (i.e., $|A_i(v)| \leq 1$), but nonetheless learns information. These nodes can be terminal (i.e., a v where $|A_i(v)| = 0$), where i learns their payoff. Or these nodes can be non-terminal, where i learns something about the history of play but has a trivial choice (i.e., a v where $|A_i(v)| = 1$).

(iii) *Nodes at which player i neither receives information nor makes a non-trivial choice:* These are nodes $v \notin N_i$. At such nodes, player i is neither active nor does i receive any information. As in the earlier setting, these nodes are included for a parsimonious model of a game tree. But they are irrelevant from i's perspective.

This alternative approach has the advantage of explicitly separating the nodes at which i receives information from those that are merely a modeling device and have no interaction with i. We will soon see that this allows us to adopt more standard definitions of perfect recall and no absentmindedness. However, it comes at the cost of not being able to simply state that H_i is a partition of the nodes V; it requires first specifying informational nodes for i. This additional notation may be worthwhile when it is important to distinguish between nodes that genuinely interact with i and those that are included only for modeling purposes.

Recall Under this definition of an extensive-form game, the conditions of no absentmindedness and perfect recall are more standard:

Definition 3.1. Say Γ^* satisfies **no absentmindedness** if the following holds: If $v \prec w$, then there is no $h \in H_i$ with $\{v, w\} \subseteq h$.

Definition 3.2. Say Γ^* satisfies **perfect recall** if, for any pair of distinct information sets $h, h' \in H_i$, the following property is satisfied: If $\{v, w\} \subseteq h$ and $v' \in h'$ with $(v', a_i, a_{-i}) \preceq v$, then there exists a node $w' \in h'$ so that $(w', a_i, a'_{-i}) \preceq w$ for some $a'_{-i} \in A_{-i}$.

How do these definitions impact the implications of recall? Proposition 2.1 requires modification:

Proposition 3.1. *If Γ^* satisfies no absentmindedness and perfect recall, then it satisfies Kuhn perfect recall. Moreover, if Γ^* is non-trivial and satisfies Kuhn perfect recall, then it satisfies perfect recall and no absentmindedness.*

Lemma 1.13 in Brandenburger (2007) provides the key step and explains the importance of non-triviality. We can see that importance in Figure 2.1. Now, P2 may or may not receive information at either of $\emptyset$ or u. In particular, suppose P2's set of information sets satisfies:

$$\{\{w, v\}, \{z_1\}, \{z_2\}, \{z_3\}, \{z_4\}\} \subseteq H_2 \subseteq \{\{\emptyset\}, \{u\}, \{w, v\}, \{z_1\}, \{z_2\}, \{z_3\}, \{z_4\}\}.$$

For any such choice of H_2, each strategy of P2 allows each non-terminal node. Since the terminal information sets are singletons, this implies that the game satisfies Kuhn perfect recall. However, whether the game satisfies perfect recall depends on whether P2 received information at u. If P2 receives information at u (i.e., if $\{u\} \in H_2$), then the game violates perfect recall. Note, when P2 receives such information, the game violates non-triviality (since $|A_2(u)| = 1$). If P2 does not receive such information (i.e., $\{u\} \notin H_2$), then the game has perfect recall. In this case, P2's only non-terminal information set has $|A_2(\{v, w\})| = 2$ and so the game is consistent with non-triviality.[1]

[1]Whether the game actually satisfies non-triviality will depend on which singletons are versus are not contained in H_1.

Proposition 2.2 remains unchanged. In particular, given two information sets for player i, viz. $h, h' \in H_i$, we can only conclude that either $S(h) \subseteq S(h')$, $S(h') \subseteq S(h)$, or $S(h) \cap S(h') = \emptyset$ if i is active at $h, h' \in H_i$. In Figure 2.1, this conclusion requires $\{u\} \notin H_2$.

Remark 3.1. At first, the requirement that i is active at both h, h' may seem surprising: There is an analogous result for Kuhn (1953) extensive for games, which does not explicitly include that qualification. However, Kuhn's definition implicitly requires that there is at most one active player at each node.

Likewise, Proposition 2.3 remains unchanged.

References

Brandenburger, A. 2007. A Note on Kuhn's Theorem. In *Interactive Logic: Selected Papers from the 7th Augustus de Morgan Workshop, London.* Vol. 1 Amsterdam Univ Pr p. 71.

Kuhn, H.W. 1953. "Extensive Games and the Problem of Information." *Contributions to the Theory of Games* 2(28):193–216.

Osborne, M.J. and A. Rubinstein. 1994. *A course in game theory.* The MIT press.