Math Camp: Lecture Notes
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Contents

1 Logic and Set Theory

1.1 Logic

A **statement** is any expression that can be true (T) or false (F). It is a custom to denote propositions by lowercase letters such as p, q, r, etc. Sometimes we write statements like $p(x)$ meaning that "x satisfies statement p". For instance, if $p(x)$ stands for "$x^2 = 4$", then $p(2)$ is true, but $p(0)$ is not.

1.1.1 Logical connectives

Logical connectives are operations on statements that produce a new statement. They are:

- $\neg$ **(negation):** *not* p, represented by $\neg p$, is the negation of p, meaning that $\neg p$ is true if p is false.
- $\vee$ **(disjunction):** p *or* q, represented by $p \vee q$, is the disjunction of p and q, meaning that $p \vee q$ is true if p is true, q is true, or both are true. (the "or" is inclusive.) For example, $(2 + 3 = 5) \vee (2 \times 3 = 6)$ is a true statement.
- $\wedge$ **(conjunction):** p *and* q, represented by $p \wedge q$, is the conjunction of p and q, meaning that $p \wedge q$ is true if p is true and q is true. For example, $(2 + 3 = 5) \wedge (2 \times 3 = 5)$ is not a true statement.

The following connectives are a bit more complex and thus, require a more detailed discussion. So before moving on to them, we summarize how the negation, disjunction, and conjunction work in a **truth table**.

p	q	$\neg p$	$p \vee q$	$p \wedge q$
T	T	F	T	T
T	F	F	T	F
F	T	T	T	F
F	F	T	F	F

Table 1. The truth table for negation, disjunction, and conjunction.

- $\to$ **(implication):** p *implies* q, represented by $p \to q$, is the implication of p and q, meaning that $p \to q$ is true unless p is true and q is false. This is perhaps the most used connective when it comes to proofs, and therefore it is crucial to have a well-understanding of it.

  Notice that, by definition, the implication $p \to q$ is true whenever p is false. This is so because an implication is true until proven false. If p is true and q is false, then the implication $p \to q$ is necessarily false. When p is false then we cannot test $p \to q$, so we continue to consider it as true. However, we will be most of the time interested in implications when we know that the hypothesis p holds true.

Remark 1.1. The implication $p \to q$ is often read as: "*p implies q*", "*if p then q*", "*p is sufficient for q*" "*p is a sufficient condition for q*", "*q if p*", "*q is necessary for p*", and "*q is a necessary condition for p*".

- $\leftrightarrow$ **(material equivalence):** p if and only if q, represented by $p \leftrightarrow q$, is the material equivalence of p and q, meaning that $p \leftrightarrow q$ is true if p and q are both true or both false.

We summarize how the implication and equivalence work in the following truth table.

p	q	$p \to q$	$p \leftrightarrow q$
T	T	T	T
T	F	F	F
F	T	T	F
F	F	T	T

Table 2. Truth table for implication, and equivalence.

1.1.2 Tautologies, contradictions, and logical equivalence

A **tautology** is a statement that is always true. A **contradiction** is a statement that is always false.

Example 1.1. The statement $p \vee \neg p$ is a tautology. The statement $p \wedge \neg p$ is a contradiction.

Say p and q are **logically equivalent** if $p \leftrightarrow q$ is a tautology. We write $p \equiv q$ or $p \iff q$.

Example 1.2. The statement p and $(\neg(\neg p))$ are logically equivalent, as the following truth table shows:

p	$\neg p$	$\neg(\neg p)$	$p \leftrightarrow (\neg(\neg p))$
T	F	T	T
F	T	F	T

Table 3. $p \equiv (\neg(\neg p))$.

Example 1.3. The statement $p \leftrightarrow (\neg p)$ is a contradiction. (Why?)

Example 1.4. The logical equivalences $(p \vee q) \equiv \neg(\neg p \wedge \neg q)$ and $(p \wedge q) \equiv \neg(\neg p \vee \neg q)$ are known as the De Morgan laws. Thus, the negations of $p \vee q$ and $p \wedge q$ are $\neg p \wedge \neg q$ and $\neg p \vee \neg q$, respectively.

Example 1.5. The statements $(p \to q)$ and $(\neg p \vee q)$ are logically equivalent. Indeed, whenever p is false, then $\neg p$ is true, which in turn implies that $\neg p \vee q$ is true. Note that in this case, $p \to q$ is true.

Now, if p is true, then $p \to q$ is false only when q is false. Observe that we reach the same conclusion for $\neg p \vee q$.

Remark 1.2. It is easy to see that $(\neg p \vee q) \equiv \neg(p \wedge \neg q)$. (Why?) Thus, the implication $p \to q$ is logically equivalent to $\neg(p \wedge \neg q)$. Notice that this implies that $p \wedge \neg q$ is the negation of $p \to q$. This logical equivalence will be important when we study proof techniques.

Example 1.6. The **contrapositive** for the implication $p \to q$ is the statement $(\neg q) \to (\neg p)$. Note that $(p \to q) \equiv ((\neg q) \to (\neg p))$. (Why?) So, the implication $p \to q$ is logically equivalent to its contrapositive $(\neg q) \to (\neg p)$. This logical equivalence will be important when we study proof techniques.

Example 1.7. The **reciprocal** of the implication $p \to q$ is the statement $q \to p$. The statements $(p \to q) \wedge (q \to p)$ and $p \leftrightarrow q$ are logically equivalent. (Why?) For this reason, "*p if and only if q*" is often expressed as "*p is a sufficient and a necessary condition for q*".

1.1.3 Quantifiers

The symbols $\forall$ ("for all", "for every", or "for each") and $\exists$ ("there exists" or "for some") are **quantifiers**. We use quantifiers to construct statements involving sets. For instance, $(\forall x \in A)(p(x))$ means that for all x in set A, the property p holds. Likewise, $(\exists x \in A)(p(x))$ states that there is some x in A for which $p(x)$ holds.

There is a subtle relationship between the quantifiers. On the one hand, note that saying that all elements in set A satisfy property p is equivalent to saying that there is no element in A for which p does not hold, so $(\forall x \in A)(p(x)) \equiv \neg(\exists x \in A)(\neg p(x))$. For example, saying that each cell phone works with batteries is exactly the same as saying that no cell phone works without batteries. Hence, the negation of $(\forall x \in A)(p(x))$ is $(\exists x \in A)(\neg p(x))$.

On the other hand, observe that saying that there is an element in set A satisfying property p is equivalent to saying that it is not true that for all elements in set A, property p does not hold, so $(\exists x \in A)(p(x)) \equiv \neg(\forall x \in A)(\neg p(x))$. To illustrate, claiming that there is a black cat is exactly the same as claiming that it is not true that all cats are not black. Hence, the negation of $(\exists x \in A)(p(x))$ is $(\forall x \in A)(\neg p(x))$.

More generally, when you negate a proposition with quantifiers, you flip each quantifier and negate the property. For example the negation of $(\forall x)(\exists y)(\forall z)(p(x, y) \wedge q(z))$ is $(\exists x)(\forall y)(\exists z)(\neg p(x, y) \vee \neg q(z))$.

A common mistake when dealing with multiple quantifiers is to change their order without realizing it. In everyday English, it is harmless because the listener's interpretation is unlikely to change. To illustrate, let $p(x, y)$ be the statement "person x is an specialist in topic y", then:

- $(\forall x)(\forall y)(p(x, y))$ states that everyone is an specialist in all topics.
- $(\exists x)(\forall y)(p(x, y))$ states that there is at least one person who is a specialist in all topics.
- $(\forall y)(\exists x)(p(x, y))$ states that given any topic, there is a specialist in it.
- $(\exists y)(\exists x)(p(x, y))$ states that there exists at least one topic in which at least one person is a specialist.

However, in daily English, we could say "there is a specialist for all topics" - $(\exists x)(\forall y)(p(x, y))$, which the listener would interpret as "in each topic, there is a specialist" - $(\forall y)(\exists x)(p(x, y))$. However, we do have the following relationships regarding the order of the quantifiers:

(i) $(\forall x)(\forall y)(p(x, y)) \equiv (\forall y)(\forall x)(p(x, y))$.

(ii) $(\exists x)(\forall y)(p(x, y))$ implies $(\forall y)(\exists x)(p(x, y))$. The reciprocal is not true. (Why?)

(iii) $(\forall y)(\exists x)(p(x, y))$ implies $(\exists y)(\exists x)(p(x, y))$. The reciprocal is not true. (Why?)

(iv) $(\exists x)(\exists y)(p(x, y)) \equiv (\exists y)(\exists x)(p(x, y))$.

1.2 Proofs

We will be mainly interested in proving statements of the form "if p then q", where p and q could be any kind of statement, involving logical connectives or quantifiers. We call p the **hypothesis** and q the **conclusion**. In this subsection, we introduce some of the basic proof strategies.

The first proof strategy is known as **direct proof**. As the name suggests, we prove the implication $p \to q$ recurring to its definition. What would we need to do to show that $p \to q$ is true? Note that one instance in which $p \to q$ is true is when p is false. Thus, to rule out the possibility that $p \to q$ is false (when p is true and q is false), we only need to show that q is true under the assumption that p is so. In short, in the direct proof strategy, we want to conclude that q holds assuming that p is true.

Example 1.8. Consider the statement: if x is even, then x^2 is even. First, let us write it using logical notation: $(\forall x \in \mathbb{Z})(x \text{ is even } \to x^2 \text{ is even})$.

To prove the statement, we first fix an arbitrary integer $x \in \mathbb{Z}$. Now, we assume that x is even, which means that there exists some integer k such that $x = 2k$. Therefore, $x^2 = 4k^2 = 2(2k^2)$ which is even since $2k^2$ is an integer. This completes the proof.

The second proof strategy we consider is known as **proof by contraposition**. By Example 1.6, we know that $p \to q$ is logically equivalent to $(\neg q) \to (\neg p)$. Thus, proving that the contrapositive is true is equivalent to proving the original implication. In short, proof by contraposition consists of direct proof of the contrapositive implication, i.e., we want to conclude that $\neg p$ holds assuming that $\neg q$ is true. The next example illustrates how convenient this proof strategy can be.

Example 1.9. Consider the statement: if x^2 is even, then x is even. First, notice that a direct proof requires us to assume that $x^2 = 2k$ for some integer k, and then show that $x = \sqrt{2k}$ takes the form of $2l$ for some integer l. This is clearly a non-trivial task.

Now, let us move on to proving the statement by contraposition. Note that in this case, we want to prove the equivalent implication $(\forall x \in \mathbb{Z})(x \text{ is odd} \to x^2 \text{ is odd})$.

Fix $x \in \mathbb{Z}$. Assume x is odd, which is to say that $x = 2k + 1$ for some integer k. As a result, $x^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$ which is an odd number since $2k^2 + 2k$ is an integer. This completes the proof.

The third proof strategy we consider is the **proof by contradiction**. By Remark 1.2, we know that the implication $p \to q$ is logically equivalent to $\neg(p \wedge \neg q)$. Thus, we can prove the implication $p \to q$ by proving that $\neg(p \wedge \neg q)$ is true. Alternatively, we could prove it by showing that $p \wedge \neg q$ cannot be true (so it is a contradiction). In short, to prove $p \to q$ by contradiction, we assume that $p \wedge \neg q$ is true, and then derive a contradiction.

Example 1.10. Consider the statement: If mn is odd, then m and n are odd. We can express the statement as $(\forall m, n \in \mathbb{Z})(mn \text{ is odd } \to (m \text{ is odd } \wedge n \text{ is odd}))$. Its negation is $(\exists m, n \in \mathbb{Z})(mn \text{ is odd } \wedge (m \text{ is even } \vee n \text{ is even}))$. Thus, we assume, as a way of contradiction, that there are some $m, n \in \mathbb{Z}$, such that mn is odd but either m or n is even. In the case that n is even, then $n = 2k$ for some integer k. Consequently, $mn = m(2k) = 2(mk)$ and since mn is an integer, then mn is even, which contradicts the initial assumption of mn being odd. In the case that m is even, we reach the same conclusion. This completes the proof.

Remark 1.3. Whenever an implication can be proven by contrapositive, it can be proven by contradiction as well. (Why?)

Remark 1.4. A proof by contradiction is usually employed when we have to prove certain statements of the form: p holds. Clearly, p is true is equivalent to $\neg p$ being false, so a way to prove that p holds is to assume that $\neg p$ is true and then derive a contradiction. The following example illustrates this.

Example 1.11. Consider the statement: $\sqrt{2}$ is not rational.

To prove it, assume by contradiction that $\sqrt{2}$ is rational, so that we can express it as a quotient $\frac{a}{b}$ in lowest terms, i.e., where a and b are integers and one of them is odd. Since

$\sqrt{2} = \frac{a}{b}$ then $a^2 = 2b^2$, so that a^2 is even. By Example 1.9, it must be that a is even. Hence, $a^2 = 4k^2$ for some integer k. This, in turn, implies that $b^2 = 2k^2$, so b^2 is even, and thus, b so is. However, this contradicts that $\frac{a}{b}$ is a quotient in lowest terms. This completes the proof.

The last proof technique is called **proof by induction**. This technique is particularly useful when we wish to prove that a property $S(n)$ holds for every positive integer. This technique requires us to prove two things:

(i) Base case: $S(1)$ is true.

(ii) Inductive step: For every $k \geq 1$, $S(k)$ implies $S(k+1)$.

Note that taken together we have: $S(1)$ is true. If $S(1)$ is true, then $S(2)$ is true. If $S(2)$ is true, then $S(3)$ is true. And so on and so forth. This proves that $S(n)$ is true for any positive integer n.

Remark 1.5. Oftentimes it is not enough to assume $S(k)$ in order to prove that $S(k+1)$ holds and the proof would work if instead we had that for any $k \geq 1$, $S(m)$ holds for every $m \leq k$. Note that changing the inductive step hypothesis by this new statement does not alter the conclusion, so we can use either one. Further, both inductive methods are equivalent. (Why?)

Example 1.12. Consider the statement: For any positive integer n, $1 + 2 + \ldots + n = \frac{n(n+1)}{2}$.

(i) Base case: In the case $n = 1$, the statement reduces to $1 = 1$, which is true.

(ii) Inductive step: Assume that the property holds for $n = k \geq 1$. Then $1 + 2 + \ldots + k + (k+1) = \frac{k(k+1)}{2} + (k+1) = \frac{(k+1)(k+2)}{2}$, so the property is true for $n = k+1$ as well.

This completes the proof.

Remark 1.6. Besides proving that $\sqrt{2}$ is not rational, we have only considered proofs of statements involving the universal quantifier. Nonetheless, we will sometimes need to prove statements involving the existential quantifier. For example, we sometimes want to disproof that the statement $(\forall x)(q(x))$, which is logically equivalent to show that $(\exists x)(\neg q(x))$.

In order to prove the statement $(\exists x)(p(x))$, we can either provide a direct example of an object satisfying the property p, or we can indirectly argue that such an object exists without being totally sure of what such an object looks like. The following examples illustrate each case.

Example 1.13. Consider the statement: There exists a real number larger than 3.

Trivially, the proof of this statement can be carried out by saying that 4 is a real number that is larger than 3.

Example 1.14. Consider the statement: There exists an irrational number x, such that $x^{\sqrt{2}}$ is a rational number. Thinking of such an object is a challenging task.

To prove it, set $x = \sqrt{2}$ which is an irrational number. If $x^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}}$ is rational, the proof is done. If not, set $x = \sqrt{2}^{\sqrt{2}}$. Then $x^{\sqrt{2}} = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{2} = 2$, which is rational. This completes the proof.

Note that we have shown that either $\sqrt{2}$ or $\sqrt{2}^{\sqrt{2}}$ satisfies the property, but we do not know which one of them.

Remark 1.7. By Example 1.7, we know that the equivalence $p \leftrightarrow q$ is logically equivalent to $(p \to q) \wedge (q \to p)$, so to prove an equivalence it suffices to prove two implications.

1.3 Basic set theory

Set, *element*, and *set membership* are some of the most intuitive notions of all mathematics, and we will consider them as primitive concepts. This means that we do not define them and rather we will get familiar with them as we work with their properties.

An apartment complex, an outfit, or a collection of numbers are examples of sets. An element of each of these sets is a unit, an item of clothing, and a certain number, respectively. Observe that a set could also be an element of another set.

Two conventions are worth noting: sets are usually written by uppercase letters such as A, B, C, etc; and elements are written by lowercase letters like a, b, c, etc. When some object x is an element of a set A, we write $x \in A$ or $A \ni x$ − although the latter is much less common. When x is not an element of A, we write $x \notin A$ or $A \not\ni x$.

A set can contain finitely or infinitely many objects. One set of special attention is the **empty set**, which contains no elements and is written by $\emptyset$ or $\{\}$. We say that two sets are **equal** if they have the same elements; if sets A and B are equal we write $A = B$; otherwise, we write $A \neq B$. For example, $\{1, 1\} = \{1\}$, $\{1, 2\} = \{2, 1\}$, and $\{\emptyset\} \neq \emptyset$. (Why?) In the case that every element in A is in B, we say that A is a **subset** of B, and we write $A \subset B$ or $B \supset A$. Clearly, $A = B$ whenever $A \subset B$ and $B \subset A$. If $A \subset B$ but $A \neq B$, then we say that A is a **proper subset of B**, which we write by $A \subsetneq B$. Formally,

Definition 1.1. Let A and B be arbitrary sets. We say that:

(i) $A = B$ if $x \in A$ if and only if $x \in B$, for all x.

(ii) $A \subset B$ if for each x, $x \in A$ implies $x \in B$.

Proposition 1.1. *Let A, B, and C be arbitrary sets. Then,*

(i) $A \subset A$.

(ii) $A \subset B$ *and* $B \subset C$, *implies* $A \subset C$.

(iii) $\emptyset \subset A$.

Proof. We only offer proof for part (iii), the remaining are left as an exercise.

First, note that we want to prove the claim $(\forall x)(x \in \emptyset \to x \in A)$. However, the statement $x \in \emptyset$ is false because the empty set has no elements. Consequently, the claim of interest is true. $\square$

Observation 1.1. *When an implication $p \to q$ is true because p is false, we say that the implication is vacuously true. Thus, claim (iii) is vacuously true.*

Definition 1.2. Let A be a set. The **power set** of A is the set $\mathcal{P}(A) = \{T \mid T \subset A\}$.

Remark 1.8. Some authors write 2^A for the power set of A. The reason is that for any nonempty sets X, Y, we write Y^X for the set of all functions from X to Y. In particular, 2^A denotes the set of all functions from A to $\{0,1\}$. This set can be understood as the set of all subsets of A. Indeed, for any $S \subset A$, define $I_S : X \to \{0,1\}$ by $I_S(x) = 1$ if $x \in A$, and 0 elsewhere. Then, for all $S \subset A$ there exists a unique function $I_S \in 2^A$, and for each function $f \in 2^A$ there exists a unique $S \subset A$ with $f = I_S$.

Example 1.15. For $A = \{1,2\}$, we have $\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, A\}$.

1.4 Set operations

Definition 1.3. Let A and B be sets.

(i) The **union** of A and B is the set $A \cup B = \{x \mid x \in A \vee x \in B\}$,

(ii) The **intersection** of A and B is the set $A \cap B = \{x \mid x \in A \wedge x \in B\}$,

(iii) The **difference** of A and B is the set $A \backslash B = \{x \mid x \in A \wedge x \notin B\}$. We usually read "$A$ *minus* B".

Proposition 1.2. *(Basic properties of union and intersection)*

Taking unions and intersections are

(i) commutative, i.e, $A \cap B = B \cap A$ and $A \cup B = B \cup A$ for any sets A, B;

(ii) associate, i.e., $A \cap (B \cap C) = (A \cap B) \cap C$ and $A \cup (B \cup C) = (A \cup B) \cup C$ for any sets A, B, C; and

(iii) distributive, i.e., $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ for any sets A, B, C.

Proof. Exercise. $\square$

Observation 1.2. *The union and intersection associative property allows one to write multiple intersections without paying attention to the parenthesis. For example, $A \cap (B \cap C)$ can be written with no confusion as $A \cap B \cap C$.*

Example 1.16. Let us offer proof for the following statement: For any set A, $A \backslash A = \emptyset$.

Note that for any x, $x \in A \backslash A$ means that $(x \in A) \wedge \neg(x \in A)$, which is a contradiction. Since $x \in \emptyset$ is also a contradiction, then $(x \in A \backslash A) \leftrightarrow (x \in \emptyset)$ holds true for any x, which means that $A \backslash A = \emptyset$.

Remark 1.9. When $A \cap B = \emptyset$, we say that A and B are **disjoint**. When there is a reference set R, the difference $R \backslash A$ is often written as A^c, which stands for **complement of** A.

Proposition 1.3. *(De Morgan's laws)*

Fix sets A, B. Then $(A \cup B)^c = A^c \cap B^c$ and $(A \cap B)^c = A^c \cup B^c$.

Proof. Let A, B be sets, and $x \in (A \cup B)^c$. This is equivalent to say that $\neg(x \in A \cup B)$, which means that $\neg(x \in A \vee x \in B)$. But this is the same as $x \notin A \wedge x \notin B$. Thus, we have $x \in A^c \wedge x \in B^c$, which is the definition of $x \in A^c \cap B^c$.

A similar argument shows the other equality. □

Remark 1.10. We will often work with sets whose elements are sets. They are usually referred to as collections or families and are denoted by script letters such as $\mathcal{A}, \mathcal{B}, \mathcal{C},$, etc.

We write $\bigcup \mathcal{A} = \bigcup_{A \in \mathcal{A}} A$, $\bigcap \mathcal{A} = \bigcap_{A \in \mathcal{A}} A$. Sometimes features of the collection $\mathcal{A}$ may change the previous notation. For instance, if $\mathcal{A} = \{A_i \mid i \in I\}$, for some index set I, we write $\bigcup \mathcal{A} = \bigcup_{i \in I} A_i$.

1.5 Relations

An **ordered pair** is an ordered list (a, b), where ordered means that for any two ordered pairs (a, b) and (c, d), it holds that $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$. In particular, (a, b) and (b, a) are generally different objects. The **Cartesian product** of the nonempty sets A and B is defined by $A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$. It is common to write A^2 for $A \times A$.

Example 1.17. If $A = \{1\}$ and $B = \{1, 2\}$, then $A \times B = \{(1, 1), (1, 2)\}$. Is $A \times B$ the same as $B \times A$?

The concept of ordered pair can be easily extended. For each positive integer n, define an **n-tuple** as an ordered list $(a_1, a_2, \ldots, a_n)$. The Cartesian product of n nonempty sets $A_1, A_2, \ldots, A_n$ is defined as $A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i\}$. The notation $\prod_{i=1}^{n} A_i$ and $\bigtimes_{i=1}^{n} A_i$ are short for $A_1 \times A_2 \times \ldots \times A_n$. Also, A^n stands for $\prod_{i=1}^{n} A$.

Let X, Y be nonempty sets. A **relation R from X to Y** is a subset of $X \times Y$. A **relation on** X is a subset of X^2. Intuitively, a relation formalizes the idea that two objects stand in a certain relationship with each other. We write xRy for $(x, y) \in R$.

Fix a relation R from X to Y. The **domain** of R is the set

$$DomR = \{x \in X \mid \exists y \in Y \text{ such that } xRy\}.$$

The **image** of R is the set

$$ImR = \{y \in Y \mid \exists x \in X \text{ such that } xRy\}.$$

The **inverse relation** of R is the relation R^{-1} from Y to X defined by $yR^{-1}x$ if and only if xRy.

Example 1.18. Let $X = \{0, 1, 2\}$. Define the relation $\geq$ on X by $(x, y) \in \geq$ if and only if $x \geq y$. Thus, $\geq = \{(2, 2), (2, 1), (2, 0), (1, 1), (1, 0), (0, 0)\}$. In addition, $Dom \geq = Im \geq = X$, and $\geq^{-1} = \leq$ on X.

Given the relations R from X to Y, and S from Y to Z, we can define a new relation called the **composition relation** of R and S from X to Z, denoted by $R \circ S$ and defined by $R \circ S = \{(x, z) \in X \times Z \mid \exists y \in Y \text{ such that } xRy \wedge ySz\}$. For a given set X, define the **identity relation** id_X on X by $id_X = \{(x, x) \in X^2 \mid x \in X\}$.

Proposition 1.4 (Properties of the composition)**.** *Let R be a relation from X to Y, S be a relation from Y to Z, and T be a relation from Z to W. Then,*

(i) $(R \circ S) \circ T = R \circ (S \circ T)$ *- the composition of relations is associative.*

(ii) $R \circ id_Y = R$ *and* $id_X \circ R = R$ *- existence of a right-identity and left-identity element.*

(iii) If $DomR = X$*, then* $id_X \subset R \circ R^{-1}$*. The reciprocal is not true in general.*

(iv) $(R \circ S)^{-1} = S^{-1} \circ R^{-1}$*.*

Proof. (i) Let $(x, w) \in X \times W$. $x((R \circ S) \circ T)w$ means that there exists $z \in Z$ such that $x(R \circ S)z$ and zTw, which in turn is equivalent to the existence of $z \in Z$ and $y \in Y$ such that xRy, ySz and zTw. This is to say that there exists $y \in Y$ such that xRy and $y(S \circ T)w$, which is the definition of $x(R \circ (S \circ T))w$.

(ii) Let $(x, y) \in X \times Y$. Note that $x(R \circ id_Y)y$ means that there exists $y' \in Y$ such that xRy' and $y' id_Y y$. Since $y' id_Y y$ holds true only when $y' = y$, then it is equivalent to saying that with xRy. This shows that $R = R \circ id_Y$.

(iii) Assume that $DomR = X$. Let $x \in X$. Since $DomR = X$, there exists $y \in Y$ such that xRy. As a result, $yR^{-1}x$, which in turn implies that $x(R \circ R^{-1})x$, and so $id_X \subset R \circ R^{-1}$.

Showing that the reciprocal is not generally true is left as an exercise.

(iv) Exercise.

$\square$

Definition 1.4. Let X be a nonempty subset and R a relation on it. We say that R is

(i) **reflexive** if xRx, for all $x \in X$.

(ii) **transitive** if xRy and yRz implies xRz, for all $x, y, z \in X$.

(iii) **symmetric** if xRy implies yRx, for all $x, y \in X$.

(iv) **antisymmetric** if xRy and yRx implies $x = y$, for all $x, y \in X$.

(v) **complete** if xRy or yRx, for all $x, y \in X$.

Example 1.19. The relation $\geq$ on $\mathbb{R}$ is reflexive, transitive, complete, and antisymmetric. Note that this relation is not symmetric, since $3 \geq 1$ but $1 \ngeq 3$.

Example 1.20. The relation $>$ on $\mathbb{R}$ is antisymmetric. Indeed, fix $x, y \in \mathbb{R}$. The statement $x > y$ and $y > x$ implies $x = y$ is vacuously true since $x > y$ and $y > x$ is a false statement.

Example 1.21. Let X be a nonempty set and $u : X \to \mathbb{R}$ be an arbitrary function. Define the relation $\succeq$ on X by $x \succeq y$ if and only if $u(x) \geq u(y)$. Note that $\succeq$ is reflexive, transitive, and complete.

Remark 1.11. Notice that every complete relation R is reflexive. Indeed, for all $x, y \in X$ we have xRy or yRx. In particular, for $x = y$ we have xRx.

1.5.1 Equivalence relations

Two special types of relations stand out: equivalence relations and orders. We say R is an **equivalence relation** if it is reflexive, symmetric, and transitive. We say R is an **order** if it is reflexive, antisymmetric, and transitive.

Example 1.22. Consider the relation R on $\mathbb{Z}$ defined by nRm if n and m have the same parity. This is an equivalence relation. Note that any integer has the same parity as itself, so R is reflexive. If n has the same parity as m, then the reciprocal also holds (symmetry). If n has the same parity as m, and m has the same parity as q, then it must be that n has the same parity as q (transitivity).

Let X be a nonempty set, a partition is a collection of disjoint nonempty subsets of X with the property that its union is X. Formally, a collection of nonempty sets $\mathcal{X}$ is a **partition** of X if: (i) for all sets $A, B \in \mathcal{X}$, if $A \neq B$ then $A \cap B = \emptyset$, and (ii) $\bigcup \mathcal{X} = X$.

Given an equivalence relation R on a nonempty set X, for any element $x \in X$ we define the **equivalence class** of x as the set $[x] = \{y \in X \mid xRy\}$. The set of equivalence classes $X/R = \{[x] \mid x \in X\}$ is called the **quotient set** of X by R.

Proposition 1.5. *Let X be a nonempty set and R an equivalence class on it. Then the quotient set X/R is a partition of X.*

Proof. We prove condition (i) by contraposition. Let $[x], [y] \in X/R$. Assume that there is some $z \in [x] \cap [y]$, which means that xRz and yRz. Since R is symmetric, we know that zRx.

Now, let us show that $[x] \subset [y]$. Notice that $w \in [x]$ means that xRw, so transitivity ensures that zRw. Once again, transitivity implies that yRw, so that $w \in [y]$.

A similar argument shows that $[y] \subset [x]$, so that $[x] = [y]$, which is what we wanted to show.

Finally, to prove condition (ii), observe that $\bigcup X/R = \bigcup_{x \in X} [x] = X$ since $x \in [x]$ by reflexivity. □

Proposition 1.5 shows that any equivalence relation produces a partition of the set X through its quotient set. In a sense, an equivalence relation partitions the set X into clusters with the property that every element within a cluster is related to each other through the equivalence relation. In short, all elements within a cluster are *equivalent* when seen through the lenses of the equivalent relation.

Example 1.23. Consider the equivalence relation R on $\mathbb{R}$ defined by xRy if $x^2 = y^2$ (verify that it is an equivalence relation). Note that for any $x \in \mathbb{R}$, $[x] = \{x, -x\}$; and that $\mathcal{R}/R = \{[x] \mid x \geq 0\}$. Thus, $\mathcal{R}/R$ can be identified with $\mathbb{R}_+ = [0, \infty)$.

Example 1.24. Consider the equivalence relation R on $\mathbb{Z}$ defined by nRm if n has the same parity as m. Note that this relation has two equivalence classes: the set of even and odd numbers.

A natural follow-up question to the connection between partitions and equivalence relations is: Can every partition of a set X produce an equivalence relation on X? The next Proposition shows that the answer is positive. Hence, equivalence relations are equivalent to partitions.

Proposition 1.6. *Let X be a nonempty subset and R a relation on it. R is an equivalence relation if and only if there exists a partition $\mathcal{X}$ of X such that, for all $x, y \in X$, xRy if and only if $x, y \in A$ for some $A \in \mathcal{X}$.* [1]

Proof. First, assume that R is an equivalence relation on X. Note that the desired partition corresponds to X/R.

Now, assume that there exists a partition $\mathcal{X}$ of X such that, for all $x, y \in X$, xRy if and only if $x, y \in A$ for some $A \in \mathcal{X}$. R is reflexive because, for any x, x is in a partition element of $\mathcal{X}$, so xRx. R is symmetric because, for any $x, y \in X$, xRy means that $x, y \in A \in \mathcal{X}$, so $y, x \in A$, which is to say that yRx. Finally, R is transitive because, for any $x, y, z \in X$, xRy

[1] For the sake of clarity, the logical notation of the proposition is (R is an equivalence relation) $\leftrightarrow$ $\{(\exists \mathcal{X}$ partition of $X)[(\forall x, y \in X)(xRy \leftrightarrow (\exists A \in \mathcal{X})(x, y \in A))]\}$.

and yRz means that there are some partition elements $A, B \in \mathcal{X}$ with $x, y \in A$ and $y, z \in B$. Since $y \in A \cap B$, it must be that $A = B$, implying that $x, z \in A$, i.e., xRz. $\square$

1.5.2 Orders

Recall that a relation R on a set X is an **order** if it is reflexive, antisymmetric, and transitive.

Example 1.25. Consider the relation $\geq$ on $\mathbb{R}^2$ defined by $(x_1, x_2) \geq (y_1, y_2)$ if $x_1 \geq y_1$ and $x_2 \geq y_2$. Then $\geq$ is an order.

Fix a nonempty set X, and an order R on it. Say $x \in X$ is an **R-maximal element** if for all $y \in X$, yRx implies $y = x$. Say $x \in X$ is an **R-minimal element** if for all $y \in X$, xRy implies $y = x$.

Proposition 1.7. *Let X be a nonempty set, and R an order on it. If X is finite, then an R-maximal and R-minimal element exists.*

Proof. Exercise. $\square$

We say R is a **complete order** if it is a complete relation and an order.

Example 1.26. The relation $\geq$ on $\mathbb{R}$ is a complete order.

Example 1.27. The relation $\geq_l$ on $\mathbb{R}^2$ defined by $(x_1, x_2) \geq_l (y_1, y_2)$ if $(x_1 > x_2)$ or $(x_1 = x_2$ and $y_1 \geq y_2)$, known as the lexicographic order, is a complete order.

Fix a nonempty set X, and a complete order R on it. Say $x \in X$ is an **R-maximum element** if xRy, for all $y \in X$. Say $x \in X$ is an **R-minimum element** if yRx, for all $y \in X$.

Proposition 1.8. *Let X be a nonempty set, and R a complete order on it. If X has an R-maximum (minimum), it is unique.*

Proof. Let $x, y \in X$ be maxima of X. Then xRz and yRz, for all $z \in X$. In particular, xRy and yRx, so $x = y$ by antisymmetry. $\square$

Proposition 1.9. *Let X be a nonempty set, and R a complete order on it. If X is finite, then the R-maximum and R-minimum elements exist.*

Proof. By induction on the number of elements of X. In the case that $X = \{x\}$, reflexivity implies that xRx, so xRy for all $y \in X$. In other words, x is the maximum. Now, assume that the proposition holds true for any set with $n \geq 1$ elements. Consider any set $X = \{x_1, \ldots, x_{n+1}\}$ with $n + 1$ elements. Notice that $X = \hat{X} \cup \{x_{n+1}\}$ where $\hat{X} = \{x_1, \ldots, x_n\}$ contains n elements. Then, the inductive hypothesis guarantees that the maximum $\hat{x}$ of $\hat{X}$ exists. Hence, by completeness $x_{n+1}R\hat{x}$ or $\hat{x}Rx_{n+1}$,

- If $x_{n+1}R\hat{x}$, then by transitivity and reflexivity, $x_{n+1}Ry$ for all $y \in X$. Thus, x_{n+1} is the maximum of X.
- If $\hat{x}Rx_{n+1}$, then $\hat{x}Ry$ for all $y \in X$. This is to say, $\hat{x}$ is the maximum of X.

This completes the proof. $\square$

1.6 Functions

Definition 1.5. Let R be a relation from X to Y. We say that R is a **function** if the following two conditions hold:

(i) For each $x \in X$, there exists $y \in Y$ such that xRy (each X element is associated with at least one Y element by R). Briefly, if $DomR = X$.

(ii) For every $x \in X$ and $y_1, y_2 \in Y$, xRy_1 and xRy_2 imply $y_1 = y_2$ (each X element is associated with at most one Y element).

In words, a relation R is a function if, for each $x \in X$, there exists a unique $y \in Y$ such that xRy.

Notation 1.1. *It is customary to denote functions by lowercase letters f, g, h, etc.*

A function f from X to Y is denoted by $f : X \to Y$. In addition, we write, with no confusion, $y = f(x)$ for xfy due to the uniqueness condition.

For any function, $f : X \to Y$, the set X is called the **domain** of f and the set Y the **codomain** of f. For any $U \subset X$, the set $f(U) = \{f(u) \in Y \mid u \in U\}$ is the **direct image** of U under f. In particular, $f(X)$ is the **image** of f. For any $V \subset Y$, the set $f^{-1}(V) = \{x \in X \mid f(x) \in V\}$ is the **inverse image** of V under f.

Definition 1.6. Let $f : X \to Y$ be a function. Say f is

(i) **injective** or **one-to-one** if $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$, for all $x_1, x_2 \in X$.

(ii) **surjective** or **onto** if $f(X) = Y$.

(iii) **bijective** if it is injective and surjective.

Proposition 1.10. *Let $f : X \to Y$ be a function, $U, U' \subset X$, and $V, V' \subset Y$. Then,*

(i) If $U \subset U'$, then $f(U) \subset f(U')$.

(ii) If $V \subset V'$, then $f^{-1}(V) \subset f^{-1}(V')$.

(iii) $U \subset f^{-1}(f(U))$. The reciprocal holds if, in addition, f is injective.

(iv) $f(f^{-1}(V)) \subset V$. The reciprocal holds if, in addition, f is onto.

Proof. We offer proof for parts (i) and (iii). The rest is left as an exercise.

(i) Assume $U \subset U'$. Let $f(x) \in f(U)$, so $x \in U \subset U'$. Hence, $f(x) \in f(U')$.

(iii) Let $x \in U$. Then $f(x) \in f(U)$, which in turn implies that $x \in f^{-1}(f(U))$.

We prove the reciprocal by contrapositive. Suppose $f^{-1}(f(U)) \not\subset U$, which means that there exists $x \in f^{-1}(f(U))$ with $x \notin U$. Since $x \in f^{-1}(f(U))$, we know that $f(x) \in f(U)$. However, $x \notin U$ implies that there exists $x' \in U$ with $x' \neq x$ and $f(x') = f(x)$, so f is not injective.

□

Proposition 1.11. *Let $f : X \to Y$ be a function, $U, U' \subset X$, and $V, V' \subset Y$. Then,*

(i) $f(U \cup U') = f(U) \cup f(U')$,

(ii) $f(U \cap U') \subset f(U) \cap f(U')$. *The reciprocal holds true if f is injective.*

(iii) $f^{-1}(V \cup V') = f^{-1}(V) \cup f^{-1}(V')$, *and* $f^{-1}(V \cap V') = f^{-1}(V) \cap f^{-1}(V')$

Proof. We offer proof for part (iii), and the rest is left as an exercise.

First, notice that $x \in f^{-1}(V \cup V')$ means that $f(x) \in V \cup V'$. This, in turn, is equivalent to $f(x) \in V$ or $f(x) \in V'$, i.e., $x \in f^{-1}(V) \cup f^{-1}(V')$.

Finally, $x \in f^{-1}(V \cap V')$ means that $f(x) \in V \cap V'$. In other words, $f(x) \in V$ and $f(x) \in V'$, which is equivalent to saying that $x \in f^{-1}(V) \cap f^{-1}(V')$. □

Note that given a function $f : X \to Y$ we can think of the *inverse relation* f^{-1}. Say f is **invertible** if f^{-1} is a function. Clearly, not all functions are invertible. For example, for $g : \mathbb{R} \to \mathbb{R}$ is defined by $g(x) = x^2$, we have that $g(1) = g(-1) = 1$, so that $(1,1) \in g^{-1}$ and $(1,-1) \in g^{-1}$. Recall that for a relation to be a function it must satisfy two conditions, namely that each element in the domain is associated with at least one element in the codomain, and that each element in the domain is associated with at most one element in the codomain.

Given $f : X \to Y$, the relation f^{-1} satisfies the first condition if for all $y \in Y$, there exists $x \in X$ such that $y = f(x)$; i.e. when f is onto. Similarly, the relation f^{-1} satisfies the second condition provided that for each $y \in Y$ and every $x, x' \in X$, $f(x) = f(x') = y$ implies $x = x'$; i.e., when f is injective. Formally,

Proposition 1.12. *Let $f : X \to Y$ be a function. f is invertible if and only if it is bijective.*

Proof. Exercise. □

Corollary 1.1. *Let $f : X \to Y$ be a function. If f is injective, then $g : X \to f(X)$ defined by $g(x) = f(x)$ is invertible.*

Proof. Exercise. □

Given two functions $f : X \to Y$ and $g : Y \to Z$, we can think of the **composition** of f and g, denoted $g \circ f$ and defined by $g \circ f = \{(x, z) \in X \times Z \mid \exists y \in Y \text{ such that } y = f(x) \wedge z = g(y)\}$ which is clearly a relation from X to Z. Moreover, the following Proposition verifies that it is a function.

Proposition 1.13. *Let $f : X \to Y$ and $g : Y \to Z$ be functions. The composition of f and g is a function from X to Z.*

Proof. Let $x \in X$, since $y = f(x) \in Y$ and g is a function, then $g(y) \in Z$ is well-defined. Thus, $x \in Dom(g \circ f)$.

Now, let $x \in X$ and $z_1, z_2 \in Z$. Assume that $x(g \circ f)z_1$ and $x(g \circ f)z_2$. This is to say, that there exists $y_1, y_2 \in Y$ satisfying $f(x) = y_1$, $f(x) = y_2$, $g(y_1) = z_1$, and $g(y_2) = z_2$. Since f is a function, it must be that $y_1 = y_2 = y$, implying that $g(y) = z_1$ and $g(y) = z_2$. Once again, since g is a function, it must be the case that $z_1 = z_2$. □

Remark 1.12. Since $g \circ f$ is a function, we can write $(g \circ f)(x)$ for the Z element that is in a relationship with x through $g \circ f$. Further, $(g \circ f)(x) = g(f(x))$.

Remark 1.13. All properties in Proposition 1.4, hold for functions as well. Two caveats are necessary: the relation id_X is a function, and all properties regarding inverse relations hold whenever the functions are invertible. Moreover, for any invertible function $f : X \to Y$, $f \circ f^{-1} = id_X$.

Proposition 1.14. *Let $f : X \to Y$ and $g : Y \to Z$ be functions. Then,*

(i) If f and g are injective, then $g \circ f$ is injective.

(ii) If f and g are onto, then $g \circ f$ is onto.

(iii) If f and g are bijective, then $g \circ f$ is bijective.

Proof. Exercise. □

1.7 Correspondences

A **correspondence** Γ from a nonempty set X to a nonempty set Y, denoted by $\Gamma : X \rightrightarrows Y$, is a function from X to $\mathcal{P}(Y)$ that assigns a nonempty subset of Y to every element of X. Hence, for all $x \in X$, $\Gamma(x) \subset Y$.

1.8 Exercises

(i) Is the statement $p \to (q \to p)$ a tautology?

(ii) Is the statement $(p \wedge q) \to \neg p$ a tautology?

(iii) Show that $(p \wedge q) \to r$ is logically equivalent to $p \to (q \to r)$.

(iv) Is $p \to (q \to r)$ logically equivalent to $q \to (p \to r)$?

(v) Show that $(p \wedge q) \vee r$ is logically equivalent to $(p \vee r) \wedge (q \vee r)$, and that $(p \vee q) \wedge r$ is logically equivalent to $(p \wedge r) \vee (q \wedge r)$.

(vi) Negate the following statements:

- $(\forall x)(\exists y)(p(x) \leftrightarrow q(y))$.
- $(\exists x)(\forall y)\{p(x,y) \to [(\exists z)q(z)]\}$.

(vii) Write out using logical notation and prove or refute that:

(a) The product of two odd numbers is odd.

(b) An integer is odd if and only if it is the sum of two consecutive integers.

(c) Two consecutive integers cannot be even.

(d) There exists a positive integer that is larger than or equal to any other positive integer.

(viii) Complete the proof of Proposition 1.1.

(ix) Let A, B be arbitrary sets. Prove that $A \subset B$ implies $\mathcal{P}(A) \subset \mathcal{P}(B)$.

(x) Prove Proposition 1.2.

(xi) Prove that for any sets A, B:

(a) $A \cap B \subset A \subset A \cup B$.

(b) $A \subset B$ if and only if $A \cap B = A$.

(c) $A \subset B$ if and only if $A \cup B = B$.

(d) $A \backslash \emptyset = A$.

(e) $A \cap (B \backslash A) = \emptyset$.

(f) $A \cup B = A \cup (B \backslash A)$.

(xii) Write $\bigcup_{k=1}^{n} B_k = B_1 \cup B_2 \cup \ldots \cup B_n$ and $\bigcap_{k=1}^{n} B_k = B_1 \cap B_2 \cap \ldots \cap B_n$. Prove by induction that for any $n \in \mathbb{N}$, we have

(a) $A \cap (\bigcup_{k=1}^n B_k) = \bigcup_{k=1}^n (A \cap B_k)$, and that $A \cup (\bigcap_{k=1}^n B_k) = \bigcap_{k=1}^n (A \cup B_k)$.

(b) $(\bigcup_{k=1}^n B_k)^c = \bigcap_{k=1}^n (B_k^c)$.

(c) $(\bigcap_{k=1}^n B_k)^c = \bigcup_{k=1}^n (B_k^c)$.

(xiii) Would your proof for the previous exercise work if we had infinitely many B sets instead of finitely many? Explain. If not, decide whether or not the properties continue to hold.

(xiv) A set-theoretic way to define the ordered pair (a, b) is $(a, b) := \{\{a\}, \{a, b\}\}$. Prove that:

(a) For every $a \neq b$, $(a, b) \neq (b, a)$.

(b) For every a, b, c, d, $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$.

(xv) Prove that $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$.

(xvi) Let A_i, B_i be nonempty sets for $i \in I$, $I \neq \emptyset$. Prove that $\left(\prod_{i\in I} A_i \bigcup \prod_{i\in I} B_i\right) \subset \prod_{i\in I}(A_i \cup B_i)$. What about the reciprocal?

(xvii) Complete the proof of Proposition 1.4.

(xviii) Is the union of reflexive relations on a set reflexive? What about intersections? Explain.

(xix) Answer the previous question considering transitive, symmetric, antisymmetric, and complete relations.

(xx) Is the composition of reflexive relations on a set reflexive?

(xxi) Answer the previous question considering transitive, symmetric, antisymmetric, and complete relations.

(xxii) Let R be a relation on a nonempty set X. Is it the following statement true? If R is symmetric and transitive then it is complete. Explain.

(xxiii) Let X be a nonempty set, and R a complete and transitive relation on it. Define the relation I on X by xIy if xRy and yRx. Prove that I is an equivalence relation.

(xxiv) Prove Proposition 1.7.

(xxv) Let R be a relation on X. Prove that if X is finite, and R is complete and transitive, then an R-maximum element exists.

(xxvi) Is the union of functions a function? What about the intersection?

(xxvii) Complete the proof of Proposition 1.10.

(xxviii) Complete the proof of Proposition 1.11.

(xxix) Prove Proposition 1.12.

(xxx) Prove Corollary 1.1.

(xxxi) Prove Proposition 1.14.

2 Real Analysis

2.1 Bounds

Fix a nonempty subset $A \subset \mathbb{R}$. The real number u is an **upper bound** of A if $u \geq a$ for all $a \in A$. The real number l is a **lower bound** of A if $l \leq a$ for all $a \in A$. Say A is **bounded above (below)** if there exists $x \in \mathbb{R}$ upper (lower) bound of A. The set A is **bounded**, if it is bounded above and bounded below.

The element $s \in \mathbb{R}$ is the **supremum** of A (denoted $\sup A$) if s is an upper bound of A and $u \geq s$ for any upper bound u of A.[2] When A is not bounded above, we say that $\sup A = \infty$. The element $r \in \mathbb{R}$ is the **infimum** of A (denoted $\inf A$) if r is a lower bound of A and $l \leq r$ for any lower bound l of A.[3] When A is not bounded below, we say that $\inf A = -\infty$.

Example 2.1. The set $I = (0, 1]$ is bounded, $\sup I = 1$ and $\inf I = 0$.

Remark 2.1. Observe that inf and sup of a set need not be elements of the set.

Example 2.2. The set $\mathbb{R}_+ = [0, \infty)$ is bounded below, is not bounded above, $\inf \mathbb{R}_+ = 0$, and $\sup \mathbb{R}_+ = \infty$.

Proposition 2.1. *Let $A \subset \mathbb{R}$ be nonempty. If $\sup A \in \mathbb{R}$, then:*

(i) $\sup A$ *is unique.*

(ii) For each $\epsilon > 0$, there exists $a \in A$ such that $\sup A - \epsilon < a$.

Proof. Let $A \subset \mathbb{R}$ and assume that $\sup A \in \mathbb{R}$. Then,

(i) Assume $s_1, s_2 \in \mathbb{R}$ are upper bounds of A, $s_1 \leq u$ and $s_2 \leq u$ for any upper bound u of A. In particular, we have $s_1 \leq s_2$ and $s_2 \leq s_1$, which implies that $s_1 = s_2$.

(ii) By contradiction. Suppose that there is $\varepsilon > 0$ such that for all $a \in A$, it holds that $\sup A - \varepsilon \geq a$. This implies that $\sup A - \varepsilon$ is an upper bound of A, a contradiction.

□

Axiom 2.1. *(Completeness axiom)*

Every nonempty subset of real numbers bounded above (below) has $\sup$ *(*$\inf$*) in* $\mathbb{R}$*.*

As a consequence of the completeness axiom we have,

Proposition 2.2. *The set of natural numbers $\mathbb{N}$ is not bounded above.*

[2] The supremum of a set is sometimes called the least upper bound.
[3] The infimum of a set is sometimes called the greatest lower bound.

Proof. By contradiction. Suppose that $\mathbb{N}$ is bounded above. Hence, the completeness axiom ensures that $a = \sup \mathbb{N}$ exists. Since for each $n \in \mathbb{N}$, it holds that $n + 1 \in \mathbb{N}$, then $n + 1 \leq a$ for each $n \in \mathbb{N}$. As a result, $n \leq a - 1$ for each $n \in \mathbb{N}$, implying that $a - 1$ is a smaller upper bound than a, a contradiction. $\square$

Remark 2.2. Proposition 2.2 is known as the Archimedean property and it is equivalent to the following statements:

(i) For all $x, y \in \mathbb{R}$ with $x > 0$, there exists $n \in \mathbb{N}$ such that $nx > y$.

(ii) For each $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$.

Proposition 2.3. *(Density of $\mathbb{Q}$ in $\mathbb{R}$) For any $x, y \in \mathbb{R}$ with $x < y$, there exists $r \in \mathbb{Q}$ such that $x < r < y$.*

Proof. Since $x < y$, the Archimedean property guarantees that there is $n \in \mathbb{N}$, such that $(y - x)n > 1$. Let $m \in \mathbb{Z}$ be such that $m - 1 \leq nx < m$. Thus, $nx < m \leq nx + 1 < ny$. Finally, $x < \frac{m}{n} < y$. $\square$

Remark 2.3. Observe that Proposition 2.3 implies that any real number can be approximated, with arbitrary accuracy, through rational numbers.

Remark 2.4. The set of rational numbers $\mathbb{Q}$ does not satisfy the completeness axiom. Indeed, the set $\{x \in \mathbb{Q} \mid 0 \leq x^2 \leq 2\}$ is bounded above but has no sup in $\mathbb{Q}$.

Definition 2.1. Let $A \subset \mathbb{R}$ be nonempty. Say $\bar{a} \in A$ is the **maximum** of A if $\bar{a} \geq a$ for all $a \in A$. Say $\underline{a} \in A$ is the **minimum** of A if $\underline{a} \leq a$ for all $a \in A$.

Proposition 2.4. *Let $A \subset \mathbb{R}$ be nonempty. Then,*

(i) $\bar{a} \in A$ is the maximum of A if and only if $\bar{a} = \sup A \in A$.

(ii) $\underline{a} \in A$ is the minimum of A if and only if $\underline{a} = \inf A \in A$.

Proof. Let $\bar{a} \in A$ be the maximum of A. Since $\bar{a} \in A$ then $\sup A \geq \bar{a}$. Now, by definition, $\bar{a}$ is an upper bound of A, so that $\sup A \leq \bar{a}$. Hence, $\bar{a} = \sup A$.

Now, assume that $\sup A \in A$. Since $\sup A$ is an upper bound of A, then $\sup A$ is the maximum of A. $\square$

Corollary 2.1. *Let $A \subset \mathbb{R}$. Then A has a maximum element if and only if $\sup A \in A$. Moreover, the maximum of A, if it exists, is unique.*

2.2 Metric spaces

Definition 2.2. A **metric** on a nonempty set X is a function

$$\begin{aligned} d : X \times X &\to \mathbb{R} \\ (x, y) &\mapsto d(x, y) \end{aligned}$$

satisfying:

(i) $d(x, y) \geq 0$, for all $x, y \in X$.

(ii) $d(x, y) = 0$ if and only if $x = y$.

(iii) $d(x, y) = d(y, x)$, for all $x, y \in X$ (Symmetry).

(iv) $d(x, z) \leq d(x, y) + d(y, z)$, for all $x, y, z \in X$ (Triangle inequality).

A **metric space** is a pair (X, d), where X is a nonempty set and d is a metric on it.

Example 2.3. The usual metric in $\mathbb{R}$ is defined by $d(x, y) = |x - y|$.

Example 2.4. The euclidean metric in $\mathbb{R}^n$ is defined by $d(x, y) = \sqrt{\sum_{k=1}^{n} (x_k - y_k)^2}$.

Example 2.5. Let X be a nonempty set and $f : X \to \mathbb{R}$. Say f is **bounded** if the set Imf is bounded. Consider the set $B(X) = \{f : X \to \mathbb{R} \mid f \text{ is bounded}\}$. Define

$$\begin{aligned} d : B(X) \times B(X) &\to \mathbb{R} \\ (f, g) &\mapsto \sup_{x \in X} |f(x) - g(x)| \end{aligned}$$

Then $(B(X), d)$ is a metric space.

Example 2.6. Let X be a nonempty set. The metric $d : X \times X \to \mathbb{R}$ defined by

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{else} \end{cases},$$

is known as the discrete metric.

Example 2.7. (Metric subspace)

If (X, d) is a metric space and $X' \subset X$, then (X', d') is a metric space where $d'(x, y) = d(x, y)$ for each $x, y \in X'$.

2.3 Sequences

Fix a metric space (X, d). A **sequence** on X is any function $x : \mathbb{N} \to X$. Sequences are usually denoted by $(x_n)_{n\geq 1}$ or $(x_n)_{n\in\mathbb{N}}$.

Definition 2.3. The sequence $(x_n)_{n\geq 1}$ **converges** to $x \in X$ if for all $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$ for all $n \geq N$. In this case x is called the **limit** of $(x_n)_{n\geq 1}$ and we write $\lim_{n\to\infty} x_n = x$, or $x_n \to x$.

Intuitively, a sequence $(x_n)_{n\geq 1}$ converges to $x \in X$ if the *tail* of the sequence approximates to its limit with any degree of accuracy. Some initial finitely many terms of the sequence may be quite apart from its limit, but eventually, all terms of the tail accumulate around it.

Proposition 2.5. *The limit of any convergent sequence is unique.*

Proof. Assume that $(x_n)_{n\geq 1}$ converges to x and also converges to y. Then for all $\varepsilon > 0$ there is a natural number N_1 such that $d(x_n, x) < \varepsilon/2$ for all $n \geq N_1$. Similarly, there is a natural number N_2 such that $d(x_m, y) < \varepsilon/2$ for all $n \geq N_2$. Now let $N = \max\{N_1, N_2\}$. Then for all $n \geq N$, we have

$$d(x, y) \leq d(x, x_n) + d(x_n, y) < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon.$$

Since this holds for all $\varepsilon > 0$, then it must be that $d(x, y) \leq 0$. Finally, it follows that $x = y$. □

Example 2.8. The sequence $(x_n)_{n\geq 1} = \left(\frac{1}{n}\right)_{n\geq 1}$ converges to 0.

Example 2.9. The sequence $(x_n)_{n\geq 1} = (n)_{n\geq 1}$ does not converge.

Definition 2.4. Say the sequence $(x_n)_{n\geq 1}$ is **bounded** if there exists $y \in X$ and $M > 0$ such that $d(x_n, y) < M$ for all $n \in \mathbb{N}$.

Proposition 2.6. *Every convergent sequence is bounded.*

Proof. Exercise. □

Remark 2.5. The reciprocal of Proposition 2.6 is not true. For example, consider the sequence of real numbers $(x_n)_{n\geq 1} = ((-1)^n)_{n\geq 1}$.

2.3.1 Sequences of real numbers

Remark 2.6. Note that the sequence of real numbers $(x_n)_{n\geq 1}$ converges to $x \in \mathbb{R}$ if and only if the sequence $(x_n - x)_{n\geq 1}$ converges to zero.

Let $(x_n)_{n\geq 1}$ be a sequence of real numbers. Say $(x_n)_{n\geq 1}$ is **increasing** if $x_n \leq x_m$ provided that $n \leq m$, for all $n, m \in \mathbb{N}$. Say $(x_n)_{n\geq 1}$ is **decreasing** if $x_n \geq x_m$ provided that $n \leq m$, for all $n, m \in \mathbb{N}$. Say $(x_n)_{n\geq 1}$ is **monotone** if it is increasing or decreasing.

Proposition 2.7. *Every monotone and bounded sequence of real numbers converges.*

Proof. Assume $(x_n)_{n\geq 1}$ is an increasing and bounded sequence of real numbers. Since $(x_n)_{n\geq 1}$ is bounded, then the set $\{x_n \mid n \geq 1\} \subset \mathbb{R}$ is bounded above, so $x = \sup\{x_n \mid n \geq 1\}$ exists by the completeness axiom. We claim that $x_n \to x$.

Fix $\epsilon > 0$, then there exists $N \in \mathbb{N}$ such that $x - \epsilon < x_N$. Since $(x_n)_{n\geq 1}$ is increasing, then we have that

$$x - \epsilon < x_N \leq x_n, \text{ provided that } n \geq N.$$

On the other side, observe that $x - \epsilon < x_n < x + \epsilon$ for $n \geq N$. Hence, $|x_n - x| < \epsilon$ for all $n \geq N$. This is to say, $x_n \to x$. □

Lemma 2.1. *(Reverse triangle inequality)*

For any $x, y \in \mathbb{R}$, it holds that $||x| - |y|| \leq |x - y|$.

Proof. Notice that $|y| = |y - x + x| \leq |y - x| + |x|$ and $|x| = |x - y + y| \leq |x - y| + |y|$. Hence, $-|x - y| \leq |x| - |y| \leq |x - y|$, which in turn implies that $||x| - |y|| \leq |x - y|$. □

Proposition 2.8. *Let $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$ be two real sequences such that $x_n \to x \in \mathbb{R}$ and $y_n \to y \in \mathbb{R}$. Then,*

(i) $|x_n| \to |x|$

(ii) $x_n + y_n \to x + y$

(iii) $x_n \cdot y_n \to x \cdot y$

(iv) If $x_n \neq 0$ for all $n \in \mathbb{N}$ and $x \neq 0$, then $\frac{y_n}{x_n} \to \frac{y}{x}$.

Proof. (i) Fix $\epsilon > 0$. Since $x_n \to x$, there exists $N \in \mathbb{N}$ such that $|x_m - x| < \epsilon$ for each $m \geq N$. Now, Lemma 2.1 implies that, for each $n \in \mathbb{N}$, $||x_n| - |x|| \leq |x_n - x|$. Thus, $||x_m| - |x|| < \epsilon$ for $m \geq N$.

(ii) Fix $\epsilon > 0$. Since $x_n \to x$ and $y_n \to y$, there exist $N \in \mathbb{N}$ such that $|x_m - x| < \frac{\epsilon}{2}$ and $|y_m - y| < \frac{\epsilon}{2}$ for each $m \geq N$. Now, the triangle inequality ensures that

$$|(x_n + y_n) - (x + y)| \leq |x_n - x| + |y_n - y| < \epsilon.$$

(iii) Fix $\epsilon > 0$. Proposition 2.6 ensures that (y_n) is a bounded sequence as it converges. This means that there is some $M \in \mathbb{N}$ such that $|y_n| \leq M$ for all $n \in \mathbb{N}$. Now, because both (x_n) and (y_n) converge, there exists a natural number N such that $|x_m - x| < \frac{\epsilon}{2M}$ and

$|y_m - y| < \frac{\epsilon}{2|x|}$.[4] As a consequence,

$$\begin{aligned} |x_n y_n - xy| &= |x_n y_n - x y_n + x y_n - xy| \\ &\leq |x_n y_n - x y_n| + |x y_n - xy| \\ &= |x_n - x||y_n| + |x||y_n - y| \\ &\leq |x_n - x| M + |x||y_n - y| \\ &< \frac{\epsilon}{2M} M + |x| \frac{\epsilon}{2|x|} = \epsilon \end{aligned}$$

□

Proposition 2.9. *(Squeeze theorem) Let $(x_n)_{n\geq 1}$, $(y_n)_{n\geq 1}$ and $(z_n)_{n\geq 1}$ be real sequences such that $x_n \leq y_n \leq z_n$ for each $n \in \mathbf{N}$. If $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = a$ then $\lim_{n\to\infty} y_n = a$.*

Proof. Exercise. □

Example 2.10. The sequence $(x_n)_{n\geq 1} = (\cos(n)/n)_{n\geq 1}$ converges to 0.

2.4 Topological notions

Fix a metric space (X, d). Let $x \in X$ and $\epsilon > 0$. Write $B(x, \epsilon)$ for the **open ball** of radius ϵ centered at x. i.e. $B(x, \epsilon) = \{y \in X \mid d(x, y) < \epsilon\}$.

Definition 2.5. Let $A \subset X$. Say $a \in A$ is an **interior point** of A if there exists $\epsilon > 0$ such that $B(a, \epsilon) \subset A$. The **interior** of A is the set $\operatorname{int} A = \{a \in A \mid a \text{ is an interior point of } A\}$.

Remark 2.7. Let $A \subset X$. Observe that $\operatorname{int} A \subset A$.

Definition 2.6. A set $F \subseteq X$ is called **open** if every point $x \in F$ is an interior point f F. In other words, if $F \subset \operatorname{int} F$.

Remark 2.8. Notice that $F \subset X$ is open if and only if $F = \operatorname{int} F$.

Example 2.11. Let (X, d) be a metric space. Any open ball is an open set. Indeed, fix $x \in X$ and $\epsilon > 0$. Let $y \in B(x, \epsilon)$ and define $\delta = \epsilon - d(x, y) > 0$. We claim $B(y, \delta) \subset B(x, \epsilon)$. Indeed, let $z \in B(y, \delta)$, meaning that $d(y, z) < \delta = \epsilon - d(x, y)$. Hence, $d(x, z) \leq d(x, y) + d(y, z) < \epsilon$. In conclusion, $z \in B(x, \epsilon)$.

Example 2.12. The set $\left[0, \frac{1}{3}\right)$ is not open in $\mathbb{R}$, but it is open in the metric space $X = [0, \frac{1}{2})$ since in this space $B\left(0, \frac{1}{3}\right) = \left[0, \frac{1}{3}\right)$.

Proposition 2.10. *Let (X, d) be a metric space. Then,*

[4] Observe that, for this proof to work, we need $x \neq 0$. How would you amend the argument for the case $x = 0$?

(i) The sets $\emptyset$ and X are open.

(ii) Arbitrary union of open sets is open.

(iii) Finite intersection of open sets is open.

Proof. (i) Directly from the definition of an open set.

(ii) Let $\mathcal{F} = \{F_i\}_{i \in I}$ be a family of open sets. Let $x \in \bigcup \mathcal{F}$, so that there exists $F_i \in \mathcal{F}$ with $x \in F_i$. Since F_i is open, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subset F_i \subset \bigcup \mathcal{F}$. Thus, x is an interior point of $\bigcup \mathcal{F}$.

(iii) Let $\mathcal{F} = \{F_i\}_{i=1}^n$ be a finite collection of open sets. Let $x \in \bigcap \mathcal{F}$, so that $x \in F_i$ for all $i = 1, \ldots, n$. Since each F_i is open, there are positive numbers $\epsilon_i > 0$ for $i = 1, \ldots, n$, such that $B(x, \epsilon_i) \subset F_i$. Let $\epsilon = \min\{\epsilon_i\}_{i=1}^n$, then $x \in B(x, \epsilon) \subset \bigcap_{i=1}^n B(x, \epsilon_i) \subset \bigcap \mathcal{F}$. □

Remark 2.9. Note that item (iii) in Proposition 2.10 cannot be extended to arbitrary intersection. Indeed, let $A_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$ for $n \in \mathbb{N}$. Then $\bigcap_{n \in \mathbb{N}} A_n = \{0\}$, which is not an open set.

Remark 2.10. Note that $F \subset X$ is open if and only if it is the union of open balls. Indeed, if F is open, then for all $x \in F$ there exists $\epsilon_x > 0$ such that $B(x, \epsilon_x) \subset F$. Hence, $\bigcup_{x \in F} B(x, \epsilon_x) = F$. The reciprocal is true in view of item (ii) of Proposition 2.10.

Proposition 2.11. *Let $A \subset X$. Then $\operatorname{int} A = \bigcup\{F \subset X \mid F \subset A \wedge F \text{ is open}\}$. In words, $\operatorname{int} A$ is the union of all open sets contained in A.*

Proof. Write $B = \bigcup\{F \subset X \mid F \subset A \wedge F \text{ is open}\}$. Let us see that $B = \operatorname{int} A$. Let $x \in B$, so that there is some open set $F \subset A$ with $x \in F$. Since F is open, there exists $\epsilon > 0$ satisfying $B(x, \epsilon) \subset F \subset A$. Hence, $x \in \operatorname{int} A$. On the other hand, if $x \in \operatorname{int} A$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subset A$. Since $B(x, \epsilon)$ is an open set contained in A, then $x \in B(x, \epsilon) \subset B$. □

Definition 2.7. A set $G \subset X$ is called **closed** if its complement $X \setminus G = G^c$ is an open set.

Example 2.13. Fix $x \in X$ and $\epsilon > 0$. Define the **closed ball** of radius ϵ centered at x as the set $B[x, \epsilon] = \{y \in X \mid d(x, y) \leq \epsilon\}$. Then, every closed ball is a closed set.

Proposition 2.12. *Let (X, d) be a metric space. Then,*

(i) X and $\emptyset$ are closed.

(ii) Finite union of closed sets is closed.

(iii) Arbitrary intersection of closed sets is closed.

Proof. Exercise. □

Remark 2.11. Note that item (ii) in Proposition 2.12 cannot be extended to infinite unions. Indeed, let $A_i = \left[-\frac{n-1}{n}, \frac{n-1}{n}\right]$ for $n \in \mathbb{N}$. The set $\bigcup_{n\in\mathbb{N}} A_i = (-1, 1)$ which is not closed.

Remark 2.12. Some sets can be both open and closed. The set X, and the empty set $\emptyset$ are both open and closed. Some sets are neither open nor closed. For example, the set $(0, 1]$ in $\mathbb{R}$. So, open is not the negation of closed, and closed is not the negation of open.

Proposition 2.13. *A set $G \subset X$ is closed if and only if for any convergent sequence $(x_n)_{n\geq 1}$ such that $x_n \in G$ for all $n \in \mathbb{N}$, we have $\lim_{n\to\infty} x_n \in G$.*

Proof. Assume $G \subset X$ is closed. Let $(x_n)_{n\geq 1}$ be a convergent sequence in G, and $x = \lim_{n\to\infty} x_n$. We need to show that $x \in G$. Assume by contradiction that $x \notin G$. Since G^c is open, there is an $\epsilon > 0$ such that $B(x, \epsilon) \subset G^c$. Now, since $\lim_{n\to\infty} x_n = x$, there exists $N \in \mathbb{N}$ such that $x_n \in B(x, \epsilon)$ for $n \geq N$. However, this is a contradiction since $x_n \in G$ for all $n \in \mathbb{N}$.

We proceed by contrapositive so as to prove the reciprocal implication. Suppose that G is not closed, i.e., that G^c is not open. Hence, there exists $x \in G^c$ such that for each $n \in \mathbb{N}$, $B\left(x, \frac{1}{n}\right) \cap G \neq \emptyset$. Let $(x_n)_{n\geq 1}$ be such that $x_n \in B\left(x, \frac{1}{n}\right) \cap G$, so that $x_n \in G$ and $\lim_{n\to\infty} x_n = x \notin G$. □

Definition 2.8. Let $A \subset X$. We say that A is **bounded** if there exists $x \in X$ and $r > 0$ such that $A \subset B(x, r)$.

Lemma 2.2. *Let $A \subset X$ be bounded. Then, for any $y \in X$, there exists $r' > 0$ such that $A \subset B(y, r')$.*

Proof. Fix $y \in X$. Since A is bounded, there exists $x \in X$ and $r > 0$ such that $A \subset B(x, r)$. Let $r' = r + d(x, y)$. If $a \in A$, then $d(a, y) \leq d(a, x) + d(x, y) < r + d(x, y) = r'$. Hence, $A \subset B(y, r')$. □

Proposition 2.14. *A finite union of bounded sets is bounded.*

Proof. Let $\mathcal{A} = \{A_i\}_{i=1}^n$ be a finite family of bounded sets. Then, for a given $y \in X$ there exist $r_1, \ldots, r_n > 0$ such that $A_i \subset B(y, r_i)$ for each $i = 1, \ldots, n$. Hence, $\bigcup \mathcal{A} \subset B(y, r)$ where $r = \max_{i=1,\ldots,n}\{r_i\}$, which means that $\bigcup \mathcal{A}$ is bounded. □

Proposition 2.15. *Let $G \subset \mathbb{R}$ be closed and nonempty. Then,*

(i) If G is bounded above, it admits a maximum.

(ii) If G is bounded below, it admits a minimum.

Proof. (i) Assume that G is bounded above. Hence, $s = \sup G$ exists. We claim that $s \in G$. To see why, assume by contradiction that $s \notin G$, so $s \in G^c$, which is an open set. Hence, there exists $\epsilon > 0$ such that $(s - \epsilon, s + \epsilon) \subset G^c$.

On the other hand, since $s = \sup G$, there exists $x \in G$ such that $s - \epsilon < x \leq s$, which implies that $x \in (s - \epsilon, s + \epsilon)$, a contradiction.

□

Definition 2.9. Let $A \subset X$.

(i) The point $x \in X$ is said to be an **adherent point** of A if $B(x, \epsilon) \cap A \neq \emptyset$, for all $\epsilon > 0$. The **closure** of A, denoted $\overline{A}$, is the set of all adherent points of A.

(ii) The point $x \in X$ is said to be a **boundary point** of A if $B(x, \epsilon) \cap A \neq \emptyset$ and $B(x, \epsilon) \cap A^c \neq \emptyset$, for all $\epsilon > 0$. The **boundary** of A, denoted ∂A, is the set of all boundary points of A.

Remark 2.13. It is clear from the definition that $\partial A \subset \overline{A}$.

Proposition 2.16. *Let $A \subset X$. Then,*

(i) $\overline{A} = \bigcap\{G \subset X \mid A \subset G \land G \text{ is closed}\}$*. In words,* $\overline{A}$ *is the intersection of all closed sets containing A. As an immediate corollary, $A \subset \overline{A}$ and $\overline{A}$ is closed.*

(ii) A is closed if and only if $A = \overline{A}$.

(iii) $\partial A = \overline{A} \setminus \operatorname{int} A$.

(iv) A is closed if and only if $\partial A \subset A$.

(v) A is open if and only if $A \cap \partial A = \emptyset$.

Proof. (i) Let $x \in \overline{A}$. Then for each $n \in \mathbb{N}$, we have $B\left(x, \frac{1}{n}\right) \cap A \neq \emptyset$. Consider any sequence $(x_n)_{n \geq 1}$ with $x_n \in B\left(x, \frac{1}{n}\right) \cap A$. Clearly, $x_n \to x$. Let $G \subset X$ be a closed set with $A \subset G$, then $x_n \in G$ for all $n \in \mathbb{N}$. Since G is closed, $x \in G$. Finally, since G was arbitrary, $x \in \bigcap\{G \subset X \mid A \subset G \land G \text{ is closed}\}$.

We proceed by contrapositive to prove the reciprocal. Let $x \notin \overline{A}$, which means that there exists $\epsilon > 0$ such that $B(x, \epsilon) \subset A^c$. Hence, $x \notin B(x, \epsilon)^c \supset \bigcap\{G \subset X \mid A \subset G \land G \text{ is closed}\}$.

(ii) We know that $A \subset \overline{A}$ and $\overline{A}$ is closed. Now, if A is closed then $A \in \{G \subset X \mid A \subset G \land G \text{ is closed}\}$, so that $\overline{A} = \bigcap\{G \subset X \mid A \subset G \land G \text{ is closed}\} \subset A$. Hence, $A = \overline{A}$.

Now, assume $A = \overline{A}$. Since $\overline{A}$ is closed, then A is closed.

(iii) Observe that $x \notin \operatorname{int} A$ means that $B(x, \epsilon) \cap A^c \neq \emptyset$. Hence, $x \in \overline{A} \setminus \operatorname{int} A$ if and only if $B(x, \epsilon) \cap A \neq \emptyset$ and $B(x, \epsilon) \cap A^c \neq \emptyset$, which is equivalent to $x \in \partial A$.

(iv) Exercise.

(v) Exercise.

$\square$

2.4.1 Completeness

Definition 2.10. Say the sequence $(x_n)_{n\geq 1}$ is a **Cauchy sequence** if for each $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $n, m \geq N$ implies $d(x_n, x_m) < \epsilon$.

Proposition 2.17. *Let $(x_n)_{n\geq 1}$ be a sequence in X. Then,*

(i) If $(x_n)_{n\geq 1}$ is a Cauchy sequence, then it is bounded.

(ii) If $(x_n)_{n\geq 1}$ is convergent, then it is a Cauchy sequence.

Proof. (i) Let $\epsilon > 0$. Since $(x_n)_{n\geq 1}$ is a Cauchy sequence, there is some $N \in \mathbb{N}$ such that $n, m \geq N$ implies $d(x_n, x_m) < \epsilon$. Fix $n_0 \geq N$. Then, for all $m \geq N$ we have $d(x_{n_0}, x_m) < \epsilon$. Thus, the set $\{x_m \mid m \geq N\}$ is bounded. Since $\{x_m \mid m < N\}$ is finite, then it is bounded. Finally, we conclude that the sequence is bounded.

(ii) Exercise.

$\square$

Remark 2.14. (Not every Cauchy sequence is convergent)

Let $X = \mathbb{R} \setminus \{0\}$. The sequence $(x_n)_{n\geq 1} = \left(\frac{1}{n}\right)_{n\geq 1}$ is a Cauchy sequence in X but has no limit in X.

Definition 2.11. Let (X, d) be a metric space. Say (X, d) is a **complete metric space** if every Cauchy sequence in X converges to a point in the space X.

Proposition 2.18. *The metric spaces $\mathbb{R}$ and $\mathbb{R}^n$ are complete.*

Example 2.14. The metric space $\mathbb{Q}$ is not complete. (Why?)

2.4.2 Compactness

Let $A \subset X$. A family $\{A_\alpha\}_{\alpha\in I}$ is a **cover** of A, if $A \subset \bigcup_{\alpha\in I} A_\alpha$. If in addition $\{A_\alpha\}_{\alpha\in I}$ is a family of open sets, we say that it is an **open cover** of A.

Definition 2.12. Let $A \subset X$. We say that A is **compact** if every open cover of A admits a finite subcover. This is to say, if $\{A_\alpha\}_{\alpha\in I}$ is an open cover of A, then there exist $\alpha_1, \ldots, \alpha_n \in I$ such that $A \subset \bigcup_{i=1}^{n} A_{\alpha_i}$.

If X is compact, we say that (X, d) is a **compact metric space**.

Example 2.15. The set $(0,1)$ is not compact in $\mathbb{R}$. Note that for the open cover $\left\{\left(\frac{1}{n},1\right)\right\}_{n=2}^{\infty}$, any finite subcover $\left\{\left(\frac{1}{n_k},1\right)\right\}_{k=1}^{j}$ we have $\bigcup_{k=1}^{j}\left(\frac{1}{n_k},1\right)=\left(\frac{1}{m},1\right)\neq(0,1)$ where $m=\max_{k=1,\dots,j}\{n_k\}$.

Proposition 2.19. *Let $A,B\subset X$. If B is compact, $A\subset B$, and A is closed, then A is compact.*

Proof. Let $A\subset B$ be closed. Let $\{A_\alpha\}_{\alpha\in I}$ be an open cover of A. Hence, $\{A_\alpha\}_{\alpha\in I}\cup\{A^c\}$ is an open cover of B. Since B is compact, there are $\alpha_1,\dots,\alpha_n\in I$ such that $B\subset\bigcup_{k=1}^{n}A_{\alpha_k}\cup A^c$. Consequently, $A=(\bigcup_{k=1}^{n}A_{\alpha_k}\cup A^c)\cap A=\bigcup_{k=1}^{n}(A_{\alpha_k}\cap A)\cup(A^c\cap A)=(\bigcup_{k=1}^{n}A_{\alpha_k})\cap A\subset\bigcup_{k=1}^{n}A_{\alpha_k}$. Therefore, A is compact. □

Proposition 2.20. *Let $A\subset X$. If A is compact, then it is bounded.*

Proof. Let $A\subset X$ be compact. Hence, $\{B(a,1)\}_{a\in A}$ is an open cover of A. Since A is compact, there are $a_1,\dots,a_n\in A$ such that $A\subset\bigcup_{k=1}^{n}B(a_i,1)$. Thus, A is bounded for it is contained in a finite union of bounded sets. □

Proposition 2.21. *Every compact metric space is complete.*

Proposition 2.22. *Let (X,d) be a metric space and $A\subset X$. If A is compact, then it is closed.*

Proof. Let $A\subset X$ be a compact set. Let $(x_n)_{n\geq1}$ be a sequence in A such that $x_n\to x\in X$. Since the metric subspace (A,d') is compact, it is complete. Since $(x_n)_{n\geq1}$ is a Cauchy sequence, then $x_n\to y\in A$. Finally, since the limits are unique, we conclude that $y=x\in A$, so that A is closed. □

Remark 2.15. Observe that we have proven that every compact set is closed and bounded. Unfortunately, the converse is not true. Indeed, let $X=\left\{\frac{1}{n}\mid n\in\mathbb{N}\right\}$ endowed with the discrete metric

$$d(n,m)=\begin{cases}1 & \text{if } n\neq m\\ 0 & \text{if } n=m\end{cases}.$$

Note that X is closed and bounded (any open ball of radius 2 contains X). However, it is not compact because any singleton is an open set (Why?) and so $\{\{x\}\}_{x\in X}$ is an open cover for X that does not admit a finite subcover.

Luckily, the converse holds in very convenient spaces.

Theorem 2.1. *(Heine-Borel) Let $A\subset\mathbb{R}^n$. A is compact if and only if it is closed and bounded.*

2.5 Functions between metric spaces

2.5.1 Continuous functions

Definition 2.13. Let $(X,d_1),(Y,d_2)$ be metric spaces and $f:X\to Y$ a function. Say f is **continuous at** $x\in X$, if given $\epsilon>0$ there exists $\delta>0$ (which may depend on ϵ and x) such that $d_1(x,x')<\delta$ implies $d_2(f(x),f(x'))<\epsilon$, for all $x'\in X$.

Say f is **continuous** if it is continuous at each point in X.

Remark 2.16. An equivalent way to express continuity at $x \in X$ is: given $\epsilon > 0$ there exists $\delta > 0$ such that $f(B(x,\delta)) \subset B(f(x),\epsilon)$.

Example 2.16. Fix $m \neq 0$. The function

$$\begin{aligned} f : \mathbb{R}^n &\to \mathbb{R}^n \\ x &\mapsto mx + b \end{aligned}$$

is continuous. It suffices to take $\delta = \frac{\epsilon}{|m|}$.

Proposition 2.23. *Let $(X,d_1),(Y,d_2)$ be metric spaces and $f : X \to Y$ a function. Then f is continuous if and only if the inverse image of any open set in Y is open in X.*

Proof. Assume f is continuous and let $F \subset Y$ be open in Y. For a given $x \in f^{-1}(F)$, we have $f(x) \in F$. Since F is open, there exists $\epsilon > 0$ such that $B(f(x),\epsilon) \subset F$. Now, since f is continuous at x, there exists $\delta > 0$ such that $f(B(x,\delta)) \subset B(f(x),\epsilon)$. Thus, $B(x,\delta) \subset f^{-1}(F)$.

Assume, on the other hand, that the inverse image of any open set in Y under f is open in X. Let $x \in X$ and let us see that f is continuous at x. Let $\epsilon > 0$. Since $f^{-1}(B(f(x),\epsilon))$ is open in X, and $x \in f^{-1}(B(f(x),\epsilon))$, there exists $\delta > 0$ such that $B(x,\delta) \subset f^{-1}(B(f(x),\epsilon))$. As a result, $f(B(x,\delta)) \subset B(f(x),\epsilon)$. □

Proposition 2.24. *(Composition of continuous functions is continuous)*

Let $(X,d_1),(Y,d_2),(Z,d_3)$ be metric spaces. If $f : X \to Y$ is continuous at x, and $g : Y \to Z$ is continuous at $f(x)$, then $g \circ f : X \to Z$ is continuous at x.

Proof. Fix $\epsilon > 0$. Since g is continuous at $f(x)$, there exists $\eta > 0$ such that $g(B(f(x),\eta)) \subset B(g(f(x)),\epsilon)$. Since f is continuous at x, then for η there exists $\delta > 0$ such that $f(B(x,\delta)) \subset B(f(x),\eta)$. Then, $(g \circ f)(B(x,\epsilon)) = g(f(B(x,\epsilon))) \subset g(B(f(x),\eta)) \subset B((g \circ f)(x),\epsilon)$. □

Proposition 2.25. *Let $(X,d_1),(Y,d_2)$ be metric spaces, and $f : X \to Y$ a function. Then f is continuous at $x \in X$ if and only if for each sequence $(x_n)_{n\geq 1}$ in X with $x_n \to x$, it holds that $f(x_n) \to f(x)$.*

Proof. Exercise. □

Corollary 2.2. *Let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ be real functions. If f and g are continuous at $x \in X$, then*

(i) $f + g$ *is continuous at* $x \in X$.

(ii) $f \cdot g$ *is continuous at* $x \in X$.

(iii) If in addition $g(x) \neq 0$, then $\frac{f}{g}$ is continuous at x.

Proof. Let $(x_n)_{n\geq 1}$ be a sequence in X with $x_n \to x$. Since f and g are continuous at x, it holds that $f(x_n) \to f(x)$ and $g(x_n) \to g(x)$. Now, Proposition 2.8 implies that $(f+g)(x_n) = f(x_n) + g(x_n) \to f(x) + g(x)$, and $(f \cdot g)(x_n) = f(x_n)g(x_n) \to f(x)g(x)$. Thus, $f+g$ and $f \cdot g$ are continuous at x. $\square$

Theorem 2.2. *(Intermediate value theorem)*

Let $f : [a,b] \to \mathbb{R}$ be a function. If f is continuous and y is a number between $f(a)$ and $f(b)$, then there exists $x^* \in [a,b]$ such that $f(x^*) = y$.

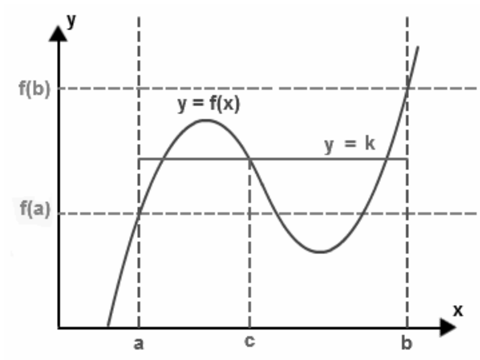


Figure 2.1 Illustration of the Intermediate Value Theorem.

Proof. Assume that $f(a) < y < f(b)$. Consider the set $A = \{x \in [a,b] \mid f(x) \leq y\}$. Note that $a \in A$ and that b is an upper bound for A. Hence, $x^* = \sup A$ exists and it is the case that $f(x^*) = y$ (Exercise). $\square$

2.5.2 Continuous functions on compact metric spaces

Proposition 2.26. *Let $(X, d_1), (Y, d_2)$ be metric spaces and $f : X \to Y$ a continuous function. If X is compact, then $f(X)$ is compact in Y.*

Proof. Let $\{F_\alpha\}_{\alpha \in I}$ be an open cover of $f(X)$ in (Y, d_2). Therefore, $f(X) \subset \bigcup_{\alpha \in I} F_\alpha$. Now, Propositions 1.10 and 1.11 imply that

$$X \subset f^{-1}(f(X)) \subset f^{-1}\left(\bigcup_{\alpha \in I} F_\alpha\right) = \bigcup_{\alpha \in I} f^{-1}(F_\alpha).$$

Since f is continuous, $f^{-1}(F_\alpha)$ is open in (X, d_1) for each $\alpha \in I$, and since X is compact, there are $\alpha_1, \ldots, \alpha_n \in I$ such that $X \subset \bigcup_{i=1}^n f^{-1}(F_{\alpha_i})$. Once again, by Propositions 1.10 and

1.11, we have that $f(X) \subset f(\bigcup_{i=1}^{n} f^{-1}(F_{\alpha_i})) = \bigcup_{i=1}^{n} f(f^{-1}(F_{\alpha_i})) \subset \bigcup_{i=1}^{n} F_{\alpha_i}$. Then, $f(X)$ is compact in (Y, d_2). □

Definition 2.14. Let $(X, d_1), (Y, d_2)$ be metric spaces, and $f : X \to Y$ a function. Say f is **bounded** if $f(X)$ is bounded in (Y, d_2).

Corollary 2.3. *Let $(X, d_1), (Y, d_2)$ be metric spaces. If $f : X \to Y$ is continuous and X is compact, then f is bounded.*

Proof. Since $f(X)$ is compact in (Y, d_2), then it is bounded in (Y, d_2). □

Definition 2.15. Let (X, d) be a metric space, and $f : X \to \mathbb{R}$ a function. Say f has a **maximum point at** $\overline{x} \in X$, if $f(\overline{x}) \geq f(x)$ for all $x \in X$. Say it has a **minimum point at** $\underline{x} \in X$, if $f(\underline{x}) \leq f(x)$ for all $x \in X$.

Corollary 2.4. *(Extreme value theorem)*

Let (X, d) be a metric space. If X is compact and $f : X \to \mathbb{R}$ is continuous, then f has a maximum and a minimum point in X.

Proof. Since $f(X)$ is compact, then $f(X)$ is closed and bounded. Consequently, Proposition 2.15 ensures that $f(X)$ admits a maximum and a minimum. This is to say, there are $\overline{x}, \underline{x} \in X$ such that f has a maximum at $\overline{x}$ and a minimum at $\underline{x}$. □

2.6 Compactness in Economics

In this section, I elaborate on the mathematical intuition of what compactness means and how it is a crucial element in Economics. Recall that a set is compact if every open cover admits a finite subcover. Thus, compactness requires a *particular* kind of finiteness - finiteness in open sets, which characterizes the notion of proximity. As a result, compactness brings some of the finite-world properties to more abstract spaces, e.g., metric spaces.

To illustrate this within Economics, let us consider an example. Suppose a decision-maker (she) must choose an alternative from the set X. Assume that her preferences are summarized by a utility function $u : X \to \mathbb{R}$, so that $u(x) \geq u(y)$ stands for "x is (weakly) preferred to y". Notice that the relation R on X defined by xRy if $u(x) \geq u(y)$ is complete and transitive (Example 1.21), which implies that an R-maximum exists provided X is a finite set (Exercise (xxv)). In other words, our decision-maker can always choose an *optimal* alternative when facing finitely many of them.

What does this have to do with compactness? Notice that Corollary 2.4 guarantees the existence of optima under the assumption of compactness and continuity. Further, in Exercise (xix) you proved that every finite set is compact. If we somehow establish that u is continuous, then our finite-setting result would be a particular case of Corollary 2.4. I will show that in two steps.

(i) Observe that any function $f : X \to \mathbb{R}$ is continuous if we endow X with the discrete metric. As a matter of fact, since every singleton is an open set (Why?), and arbitrary unions of open sets are open, then every subset of X is open. Hence, f pulls open sets in $\mathbb{R}$ back into open sets in X, which means that f is continuous. In particular, u is continuous provided that X is endowed with the discrete metric.

However, there are myriad more metrics we could think of. So, how robust is the continuity of u to changes in the metric? The next step shows that our conclusion does not depend on the particular metric when X is a finite set.

(ii) First, let us introduce the notion of **equivalent metrics**, which formalizes the idea of two metrics being *basically the same.*

Definition 2.16. Let X be a nonempty set, and $d_1, d_2 : X \times X \to \mathbb{R}$ two metrics on X. Say d_1 and d_2 are **equivalent** if for every $A \subset X$, A is d_1-open if and only if it is d_2-open.

Example 2.17. Let $X = \mathbb{R}^n$. The Euclidean metric $d_1(x, y) = \sqrt{\sum_i (x_i - y_i)^2}$ and the metric of the maximum $d_2(x, y) = \max_i\{|x_i - y_i|\}$ are equivalent.

Example 2.18. Let $X = \mathbb{R}$. The usual metric $d_1(x, y) = |x - y|$ and the discrete metric

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{else} \end{cases},$$

are not equivalent.

Proposition 2.27. *Let X be a nonempty set, and $d_1, d_2 : X \times X \to \mathbb{R}$ two metrics on X. Let $(x_n)_{n\geq 1}$ be a sequence on X, $x \in X$, and $f : X \to \mathbb{R}$ a function. If d_1 and d_2 are equivalent, then*

(a) $x_n \xrightarrow{d_1} x$ if and only if $x_n \xrightarrow{d_2} x$, and

(b) f is d_1-continuous if and only if it is d_2-continuous.

Proof. (a) Exercise.

(b) Assume f is d_1-continuous. Let $V \subset \mathbb{R}$ be open, so $f^{-1}(V)$ is d_1-open. This implies that $f^{-1}(V)$ is d_2-open given that d_1 and d_2 are equivalent metrics. Thus, f is d_2-continuous.

□

Proposition 2.28. *Let X be a finite nonempty set. Then, all metrics on X are equivalent to each other.*

Proof. Let d be an arbitrary metric on X. For each $x \in X$, define $r_x = \min_{y \neq x}\{d(x, y)\}$, which is well-defined and positive as X is finite. Hence, $\{x\} = B(x, r_x)$ is an open set. Thus, every subset of X is an open set as it can be written as the union of open sets.

Since d was arbitrary, then every subset of X is an open set regardless of the metric, which implies that all metrics are equivalent on X. □

Steps (i) and (ii) show that u is continuous regardless of the metric under consideration on the condition that X is finite. Thus, the finite decision-problem result is embedded in Corollary 2.4.

Now, we turn to the case when X has infinitely many elements. What conditions would ensure that we can find an optimal alternative to pick given preferences $u : X \to \mathbb{R}$? The answer is to assume that X is compact and u is continuous. If we get rid of one of those assumptions, the decision problem may not be well-defined.

Example 2.19. Consider $X = (0, 1)$, and $u(x) = \frac{1}{x}$. Observe that u is continuous, X is bounded but not closed (and hence, not compact), and that this decision problem does not have a solution.

Example 2.20. Consider $X = \mathbb{R}$, and $u(x) = x$. Observe that u is continuous, X is closed but unbounded (and hence, not compact), and that this decision problem does not have a solution.

Example 2.21. Consider $X = [0, 1]$, and

$$u(x) = \begin{cases} x & \text{if } x < 1 \\ 0 & \text{else} \end{cases}.$$

Observe that X is compact but u is not continuous, and that this decision problem does not have a solution.

Example 2.22. By Theorem 2.1, a set in $\mathbb{R}^n$ is compact if and only if it is closed and bounded. Unfortunately, this equivalence does not always hold. This example presents an ill-defined decision problem in which the utility function is continuous, the space is closed and bounded, but not compact.

Consider $X = \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$ endowed with the discrete metric, and $u(x) = -\frac{1}{x}$. Observe that:

(i) X is closed and bounded because $X \subset B(1, 2)$,

(ii) X is not compact as $\{\{x\} \mid x \in X\}$ is an open cover that does not admit a finite subcover,

(iii) u is continuous, and

(iv) There is no solution to the decision problem.

To sum up, compactness is a must in the study of decision problems, which are ubiquitous in Economics. A well-understanding of its importance will be rewarding during the first-year courses.

2.7 Differentiation

This section is focused on the differentiation of univariate functions, that is, functions that map some subset of the real line to the reals. We also briefly present some definitions and a key result about the differentiation of multivariate functions for future reference.

Definition 2.17. Let f be a real function defined around $a \in \mathbb{R}$ (not necessarily at a). Say that the **limit of** f **as** x **approaches to** a **is** L, denoted $\lim_{x\to a} f(x) = L$, if for each $\epsilon > 0$ there exists $\delta > 0$ such that $0 < |x - a| < \delta$ implies $|f(x) - L| < \epsilon$.

Example 2.23. Let us prove that $\lim_{x\to 1}(2x + 1) = 3$. Fix $\epsilon > 0$. Consider $\delta = \frac{\epsilon}{2} > 0$. Assume that $0 < |x - 1| < \delta = \frac{\epsilon}{2}$, then

$$|(2x + 1) - 3| = 2|x - 1| < 2\delta = \epsilon.$$

Proposition 2.29. *Let $f : \mathbb{R} \to \mathbb{R}$. Then, f is continuous at $a \in \mathbb{R}$ if and only if $\lim_{x\to a} f(x) = f(a)$.*

Proof. By definition. ☐

Proposition 2.30. *Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be functions. If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, then*

(i) $\lim_{x\to a}(f + g)(x) = L + M$.

(ii) $\lim_{x\to a}(f \cdot g)(x) = L + M$.

(iii) If in addition $M \neq 0$, then $\lim_{x\to a} \frac{f}{g}(x) = \frac{L}{M}$.

Proof. Exercise. ☐

Proposition 2.31. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function. If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} f(x) = M$, then $L = M$.*

Proof. Exercise. ☐

Let $I = (a, b)$ be an open interval, and let $f : I \to \mathbb{R}$. Say f is **differentiable** at $x \in I$ if the limit

$$\lim_{h\to 0} \frac{f(x + h) - f(x)}{h} = f'(x).$$

exists. When it does, the number $f'(x)$ is called the **derivative** of f at x. If f is differentiable at every point of I, then f is said to be **differentiable** on I, call $f' : I \to \mathbb{R}$ the **derivative** of f. If f' is also differentiable, then f is said to be twice differentiable, and f'' is called the **second derivative** of f. Some functions are infinitely differentiable.

The derivative of a function f at a point x can be interpreted as the slope of the tangent line to the graph of f at the point $(x, f(x))$. The derivative of a function f at a point x can also be interpreted as the rate of change of f at x. To see this, write $\Delta x = h$ for some small $h > 0$ and write $\Delta y = f(x+h) - f(x)$. Then, notice that the rate of change is

$$\frac{\Delta y}{\Delta x} = \frac{f(x+h) - f(x)}{h}.$$

For this reason, the derivative is often described as the **instantaneous rate of change** of f at x. The following proposition shows that differentiability is a stronger condition than continuity.

Proposition 2.32. *Fix $f : (a, b) \to \mathbb{R}$. If f is differentiable at $x \in (a, b)$, then f is continuous at x.*

Proof. Let $f : (a, b) \to \mathbb{R}$ be differentiable at $x \in (a, b)$. Then, $f'(x)$ exists. Since $f'(x)$ exists, we know that

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x).$$

Hence,

$$\begin{aligned}
\lim_{h \to 0} [f(x+h) - f(x)] &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} h. \\
&= \left(\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \right) (\lim_{h \to 0} h) \\
&= f'(x) \\
&= 0
\end{aligned}$$

Thus, f is continuous at x. $\square$

Not all continuous functions are differentiable. For instance, the function $f(x) = |x|$ is continuous on $\mathbb{R}$, but is not differentiable at 0. To see why, write

$$g(h) = \frac{f(x+h) - f(x)}{h}.$$

So, to show that f is not differentiable at 0, we need to show that $\lim_{h \to 0} g(h)$ does not exist. To do so, let x_n be a sequence of positive numbers that converges to 0. Then, $g(x_n) = 1$ for all n. Now, let y_n be a sequence of negative numbers that converges to 0. Then, $g(y_n) = -1$

for all n. Hence,

$$\lim_{n\to\infty} g(x_n) = 1 \quad \text{and} \quad \lim_{n\to\infty} g(y_n) = -1.$$

Since these sequential limits are different, and the limit of functions is unique, it follows that $\lim_{h\to 0} g(h)$ does not exist. Therefore, f is not differentiable at 0.

Example 2.24. The function $f(x) = x^2$ is differentiable and has derivative $f'(x) = 2x$.

To see it, fix a and consider the function $f(x) = x^2$. Then,

$$g(h) = \frac{f(a+h) - f(a)}{h} = \frac{(a+h)^2 - a^2}{h} = \frac{a^2 + 2ah + h^2 - a^2}{h} = \frac{2ah + h^2}{h}.$$

Let $h \to 0$. Then, $g(h) \to 2a$. Therefore, $f'(a) = 2a$.

2.7.1 Properties of differentiation

This link provides a nice cheat sheet to compute derivatives of standard functions. These notes do not attempt to rigorously prove all the results provided in the link above. Instead, we focus on some of the most important tools.

Proposition 2.33. *Fix $I = (a, b)$. Let $f, g : I \to \mathbb{R}$ be differentiable on I, then for every $x \in I$ we have*

(i) $f + g$ is differentiable on I and $(f + g)'(x) = f'(x) + g'(x)$.

(ii) $f \cdot g$ is differentiable on I and $(f \cdot g)'(x) = f(x) \cdot g'(x) + g(x) \cdot f'(x)$.

(iii) If $g(x) \neq 0$, then f/g is differentiable on I and $(\frac{f}{g})'(x) = \frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{g^2(x)}$.

Theorem 2.3. *(Rolle's theorem)*

Let $f : [a, b] \to \mathbb{R}$ be a differentiable function such that $f(a) = f(b)$. Then, there exists $c \in (a, b)$ such that $f'(c) = 0$.

Theorem 2.4. *(Mean value theorem)*

Let $f : (a, b) \to \mathbb{R}$ be a differentiable function. Then, there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof. The proof follows from Rolle's theorem. To see why, define $g(x) = f(x) - \frac{f(b)-f(a)}{b-a}(x-a)$. Then, g is differentiable and $g(a) = g(b)$. By Rolle's theorem, there exists $c \in (a, b)$ such that $g'(c) = 0$. But, $g'(c) = f'(c) - \frac{f(b)-f(a)}{b-a} = f'(c) - \frac{f'(c)(b-a)}{b-a} = 0$, which implies that $f'(c) = \frac{f(b)-f(a)}{b-a}$. $\square$

The geometric intuition of the mean value theorem is that if f is a differentiable function on an interval (a, b), then there exists $c \in (a, b)$ such that the tangent to f at c is parallel to the secant joining $f(a)$ and $f(b)$. See Figure 2.2.

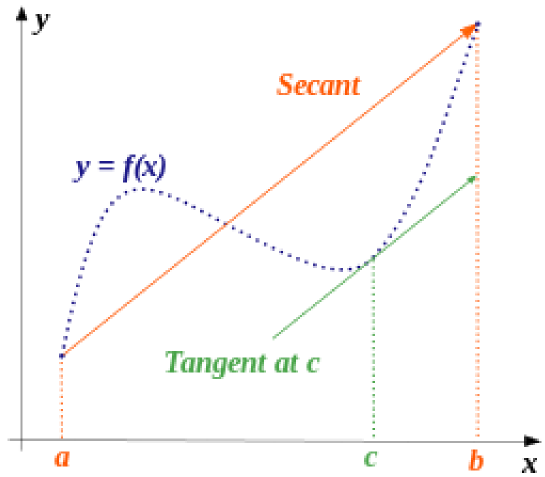


Figure 2.2 Illustration of the mean value theorem

2.7.2 Differentiable monotone functions

Fix $S \subset \mathbb{R}$. Call a function $f : S \to \mathbb{R}$ **increasing** if $x < y$ implies $f(x) \leq f(y)$. Call $f : S \to \mathbb{R}$ **strictly increasing** if $x < y$ implies $f(x) < f(y)$. Call a function $f : S \to \mathbb{R}$ **decreasing** if $x < y$ implies $f(x) \geq f(y)$. Call $f : S \to \mathbb{R}$ **strictly decreasing** if $x < y$ implies $f(x) > f(y)$.

Proposition 2.34. *Fix an open interval $I = (a, b)$. Let $f : I \to \mathbb{R}$ be differentiable. Then, the following hold.*

(i) $f'(x) \geq 0$ for all $x \in I$ if and only if f is increasing.

(ii) If $f'(x) > 0$ for all $x \in I$ then f is strictly increasing.

(iii) $f'(x) \leq 0$ for all $x \in I$ if and only if f is decreasing.

(iv) If $f'(x) < 0$ for all $x \in I$ then f is strictly decreasing.

The converse of (ii) and (iv) do not hold, as the following example illustrates.

Example 2.25. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^3$. Then, $f'(x) = 3x^2$. Notice that f is strictly increasing even though $f'(x) = 0$ for $x = 0$.

Lemma 2.3. *Fix an open interval $I = (a, b)$. If f is strictly monotone, then it is injective.*

Proof. Suppose f is strictly monotone on I. Then, for all $x, y \in I$ we have that $x < y$ implies $f(x) < f(y)$ or $x > y$ implies $f(x) > f(y)$. Hence, if $f(x) = f(y)$, then $x = y$. $\square$

Theorem 2.5. *Let $I = (a, b)$ be an open interval and let $f : I \to \mathbb{R}$ be strictly monotone, continuous, and differentiable on (a, b) with $f'(x) \neq 0$ for each $x \in (a, b)$. Then, f has an inverse $f^{-1} : (c, d) \to I$, where $(c, d) = f(I)$. Furthermore, f^{-1} is continuous and differentiable on (c, d), and $f^{-1'} = (\frac{1}{f'})$.*

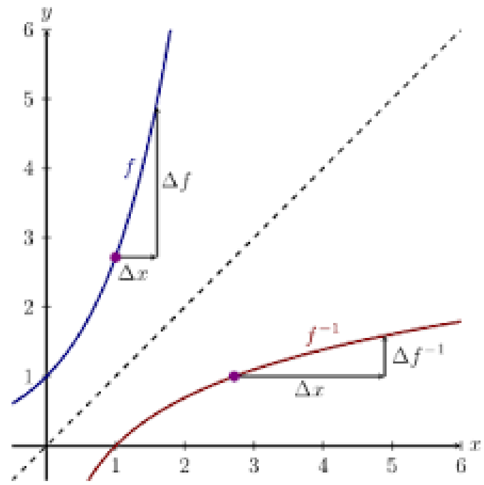


Figure 2.3 Illustration of the inverse value theorem. The slope of the inverse function can be obtained by taking the mirror of f along the line $y = x$ and then taking the slope of the mirror image.

Proof. Since f is strictly monotone, by Lemma 2.3, f is injective. Hence, there exists a function $f^{-1} : (c, d) \to I$ such that $f \circ f^{-1} = id$, where $(c, d) = f(I)$. Fix $x_0 \in (a, b)$ and write $y_0 = f(x_0)$. Then,

$$\begin{aligned}\frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} &= \frac{x - x_0}{f(x) - f(x_0)} \\ &= \frac{1}{\left(\frac{f(x)-f(x_0)}{x-x_0}\right)}\end{aligned}$$

So, because f is differentiable at x_0, it follows that

$$\begin{aligned}\lim_{y\to y_0} \tfrac{f^{-1}(y)-f^{-1}(y_0)}{y-y_0} &= \lim_{x\to x_0} \frac{1}{\left(\frac{f(x)-f(x_0)}{x-x_0}\right)} \\ &= \tfrac{1}{f'(x_0)}.\end{aligned}$$

Since f^{-1} is differentiable at every point of (c, d), it follows that f^{-1} is differentiable on (c, d). □

2.7.3 Partial derivatives

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function. The **partial derivative** of f with respect to its argument x_j at $a \in \mathbb{R}^n$ is defined as

$$\frac{\partial f}{\partial x_j}(a) = \lim_{h\to 0} \frac{f(a + he_j) - f(a)}{h}$$

whenever the limit exists.

Remark 2.17. The vector $e_j \in \mathbb{R}$ is defined as the vector whose all components are zero, and the $j-$th component is 1. For example, in $\mathbb{R}^2$, $e_1 = (1,0)$ and $e_2 = (0,1)$.

Notation 2.1. *It is common to write f_j or f_{x_j} for $\frac{\partial f}{\partial x_j}$.*

Remark 2.18. In economics, we usually write multivariate functions in terms of a variable of interest and the rest of them. For instance, $f(x_1, \ldots, x_n)$ can be written $f(x_j, x_{-j})$ where x_{-j} denotes all remaining variables other than x_j. This allows us to focus on important variables.

Hence, using this notation we could equivalently write

$$\frac{\partial f}{\partial x_j}(a) = \lim_{h \to 0} \frac{f(a_j + h, a_{-j}) - f(a)}{h}.$$

Example 2.26. Consider the function $f(x,y) = 2x^2 + y$. Then

$$\begin{aligned}
f_x(1,1) &= \lim_{h \to 0} \frac{f((1,1) + h(1,0)) - f(1,1)}{h} \\
&= \lim_{h \to 0} \frac{f(1+h,1) - f(1,1)}{h} \\
&= \lim_{h \to 0} \frac{2(1+h)^2 + 1 - (2+1)}{h} \\
&= \lim_{h \to 0} \frac{2h(2+h)}{h} \\
&= \lim_{h \to 0} 2(2+h) \\
&= 4.
\end{aligned}$$

Remark 2.19. Every partial derivative can be understood as the derivative of a univariate function. Indeed, let $f : \mathbb{R}^n \to \mathbb{R}$ be a function. Suppose that we are interested in computing $f_{x_j}(a)$ for $a \in \mathbb{R}^n$.

Consider the auxiliary function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = f(x, a_{-j})$. Then,

$$\begin{aligned}
g'(a_j) &= \lim_{h \to 0} \frac{g(a_j + h) - g(a_j)}{h} \\
&= \lim_{h \to 0} \frac{f(a_j + h, a_{-j}) - f(a)}{h} \\
&= \frac{\partial f}{\partial x_j}(a).
\end{aligned}$$

Thus, all properties and interpretations of univariate derivatives hold for partial derivatives. In particular,

Proposition 2.35. *Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be functions. If f_{x_j} and g_{x_j} exist on $\mathbb{R}^n$, then for every $x \in I$ we have*

(i) $(f+g)_{x_j}(x) = f_{x_j}(x) + g_{x_j}(x)$.

(ii) $(f \cdot g)_{x_j}(x) = f(x) \cdot g_{x_j}(x) + g(x) \cdot f_{x_j}(x)$.

(iii) *If* $g(x) \neq 0$, $(\frac{f}{g})_{x_j}(x) = \frac{f_{x_j}(x) \cdot g(x) - f(x) \cdot g_{x_j}(x)}{g^2(x)}$.

Remark 2.20. The concept of differentiability for real functions is equivalent to derivability. Unfortunately, this is not the case for multivariate functions. For the latter, differentiability is a stronger concept than derivability, meaning that all differentiable functions are derivable and that there are functions for which all partial derivatives are well-defined but the functions are not differentiable. Hence, although some of the results on differentiability for univariate functions extend to multivariate functions, a rigorous treatment requires additional tools that are beyond the scope of these notes.

Notation 2.2. *The vector of all partial derivatives of* f *is referred to as the gradient of* f, *and denoted by* $\nabla f(x) = (f_i(x))_{i=1}^n$.

Notation 2.3. *The matrix of all second-order partial derivatives of* f *is referred to as the Hessian matrix of* f, *and denoted by* $H_f(x) = (f_{ij}(x))_{i,j=1}^n$.

Theorem 2.6. *(Young's theorem) Let* $X \subset \mathbb{R}^n$ *be open, and* $f : X \to \mathbb{R}$ *a function. If* f *is twice continuously differentiable on* X, *then* $f_{ij} = f_{ji}$, *for all* $i, j = 1, \ldots, n$.

Remark 2.21. If $f : X \to \mathbb{R}$ is twice continuously differentiable, then $H_f(x)$ is a symmetric matrix.

2.8 Exercises

(i) Let $X, Y \subset \mathbb{R}$ be nonempty and bounded above. Prove that $X \subset Y$ implies $\sup X \leq \sup Y$.

(ii) Let $X, Y \subset \mathbb{R}$ be nonempty and bounded below. Prove that $X \subset Y$ implies $\inf X \geq \inf Y$.

(iii) Prove that for each $x \in \mathbb{R}$, and any $\epsilon > 0$, there exists $r \in \mathbb{Q}$ such that $x - \varepsilon < r < x + \varepsilon$.

(iv) Let $x, y \in \mathbb{R}$. Prove that if $x < y + \epsilon$ for all $\epsilon > 0$, then $x \leq y$.

(v) Prove Proposition 2.6.

(vi) Let (X, d) be a metric space, $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$ be sequences on X. Prove that if, for some $M \in \mathbb{N}$, $y_n = x_n$ for all $n \geq M$, and (x_n) converges, then (y_n) converges.

(vii) Prove Proposition 2.9

(viii) Show that $(x_n)_{n\geq 1} = (\cos(n)/n)_{n\geq 1}$ converges to 0.

(ix) Let $A \subset X$. Prove that $\operatorname{int} A$ is the largest open set contained in A, i.e., for any open set $F \subset X$, $F \subset A$ implies $F \subset \operatorname{int} A$.

(x) Prove that every closed ball is a closed set.

(xi) Prove Proposition 2.12

(xii) Let (X, d) be a metric space. Prove that any finite set in X is closed.

(xiii) Let (X, d) be a metric space. Prove that any finite set in X is bounded.

(xiv) Let $A \subset X$. Prove that:

(a) $\overline{A} = (\text{int}\,(A^c))^c$, and

(b) $\partial A = \overline{A} \cap \overline{A^c}$.

(xv) Finish the proof of Proposition 2.16.

(xvi) Let $A \subset X$. Prove that $\overline{A}$ is the smallest closed set containing A. That is, for any closed set $G \subset X$, $A \subset G$ implies $\overline{A} \subset G$.

(xvii) Let (X, d) be a metric space, and $A, B \subset X$. Prove that:

(a) $\text{int}\,A \cap \text{int}\,B = \text{int}\,(A \cap B)$.

(b) $\text{int}\,A \cup \text{int}\,B \subset \text{int}\,(A \cup B)$. The reciprocal is not true. Find a counterexample.

(c) $\overline{(A \cap B)} \subset \overline{A} \cap \overline{B}$. The reciprocal is not true. Find a counterexample.

(xviii) Complete the proof of Proposition 2.17.

(xix) Let (X, d) be a metric space. Prove that any finite set is compact.

(xx) Let (X, d) be a metric space. Prove that the finite union of compact sets is compact. How about infinite unions?

(xxi) Let $A, B \subset X$. Prove that if A is closed and B is compact, then $A \cap B$ is compact.

(xxii) Prove Proposition 2.25.

(xxiii) Complete the proof of Theorem 2.2.

(xxiv) Complete the proof of Proposition 2.27.

(xxv) Prove Proposition 2.30.

(xxvi) Prove Proposition 2.31.

3 Linear Algebra

This chapter will focus on some of the basic concepts of linear algebra. This set of notes concerns the n-dimensional Euclidean space $\mathbb{R}^n$, and the notation and treatment are econometric-oriented.

An element of $\mathbb{R}^n$ is called a **vector**. More specifically, a vector $\mathbf{x} \in \mathbb{R}^n$ is an ordered list of numbers denoted by

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Notation 3.1. *The vector* $\mathbf{x}$ *is often written as* $(x_1, \ldots, x_n)'$ *or simply* (x_i).

Say two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ are **equal** if $x_i = y_i$ for each $i = 1, \ldots, n$. There are two fundamental operators in $\mathbb{R}^n$: **addition** and **scalar multiplication**. Fix a vector $\mathbf{x} = (x_1, \ldots, x_n)' \in \mathbb{R}^n$, a vector $\mathbf{y} = (y_1, \ldots, y_n)' \in \mathbb{R}^n$ and a real number $\lambda \in \mathbb{R}$. Write $\mathbf{x} + \mathbf{y} = (x_1 + y_1, ..., x_n + y_n)'$ for the **sum** of the vectors $\mathbf{x}$ and $\mathbf{y}$. Write $\lambda\mathbf{x} = (\lambda x_1, ..., \lambda x_n)'$ for the vector $\mathbf{x}$ **multiplied by the scalar** λ.

3.1 Orthogonal vectors

Definition 3.1. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. The **dot product** of $\mathbf{x}$ and $\mathbf{y}$ is $\mathbf{x} \cdot \mathbf{y} = \sum_{k=1}^{n} x_k y_k \in \mathbb{R}$.

Write $\mathbf{0}$ for the zero vector. i.e., $\mathbf{0} = (0, \ldots, 0)' \in \mathbb{R}^n$.

Proposition 3.1. *Let* $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$ *and* $\lambda \in \mathbb{R}$. *Then,*

(i) $\mathbf{x} \cdot \mathbf{x} \geq 0$.

(ii) $\mathbf{x} \cdot \mathbf{x} = 0$ *if and only if* $\mathbf{x} = \mathbf{0}$.

(iii) $\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$.

(iv) $\mathbf{x} \cdot (\mathbf{y} + \mathbf{z}) = \mathbf{x} \cdot \mathbf{y} + \mathbf{x} \cdot \mathbf{z}$.

(v) $(\lambda\mathbf{x}) \cdot \mathbf{y} = \lambda(\mathbf{x} \cdot \mathbf{y})$.

(vi) $(\mathbf{x} + \mathbf{y}) \cdot (\mathbf{x} + \mathbf{y}) = \mathbf{x} \cdot \mathbf{x} + 2(\mathbf{x} \cdot \mathbf{y}) + \mathbf{y} \cdot \mathbf{y}$.

Proof. Exercise. □

Definition 3.2. Let $\mathbf{x} \in \mathbb{R}^n$. The **norm** of $\mathbf{x}$ is $\|\mathbf{x}\| = \sqrt{\mathbf{x} \cdot \mathbf{x}}$.

Remark 3.1. Note that for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$. Thus, $\|\mathbf{x}\| = d(\mathbf{x}, \mathbf{0})$. Hence, the norm of $\mathbf{x}$ is also known as the length of $\mathbf{x}$.

Proposition 3.2. *Let* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, *and* $\lambda \in \mathbb{R}$. *Then,*

(i) $\|\mathbf{x}\| \geq 0$.

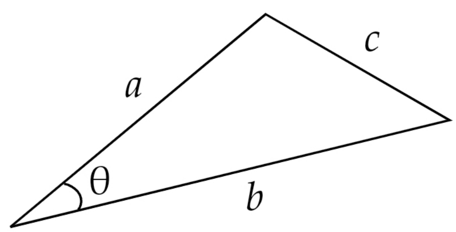
(ii) $\|\mathbf{x}\| = 0$ *if and only if* $\mathbf{x} = \mathbf{0}$.

(iii) $\|\lambda \mathbf{x}\| = |\lambda| \|\mathbf{x}\|$.

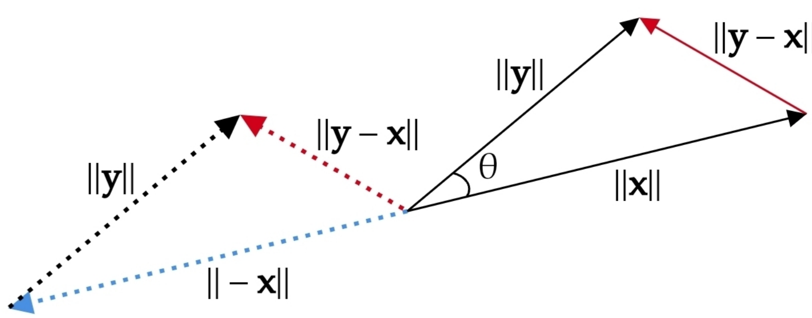
(iv) $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$.

Proof. Exercise. □

Consider the following triangle



Then, the law of cosines establishes that $c^2 = a^2 + b^2 - 2ab\cos(\theta)$. Now, for any nonzero vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we can form a triangle as the following figure depicts



Thus, applying the law of cosines we obtain

$$\|\mathbf{y} - \mathbf{x}\|^2 = \|\mathbf{y}\|^2 + \|\mathbf{x}\|^2 - 2\|\mathbf{y}\|\|\mathbf{x}\|\cos(\theta)$$
$$\|\mathbf{y}\|^2 + \|\mathbf{x}\|^2 - 2(\mathbf{y} \cdot \mathbf{x}) = \|\mathbf{y}\|^2 + \|\mathbf{x}\|^2 - 2\|\mathbf{y}\|\|\mathbf{x}\|\cos(\theta)$$

Consequently, the angle θ formed by the nonzero vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ satisfies

$$\cos(\theta) = \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|}.$$

Thus, $\mathbf{x}$ and $\mathbf{y}$ are perpendicular ($\theta = 90^\circ$, so $\cos(\theta) = 0$) if and only if $\mathbf{x} \cdot \mathbf{y} = 0$.

Definition 3.3. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, and $X \subset \mathbb{R}^n$.

(i) Say $\mathbf{x}$ and $\mathbf{y}$ are **orthogonal** if $\mathbf{x} \cdot \mathbf{y} = 0$.

(ii) Say $\mathbf{x}$ is **orthogonal to the set** X if $\mathbf{x}$ is orthogonal to each element of X.

Remark 3.2. Observe that the definition of orthogonality implies that $\mathbf{0} \in \mathbb{R}^n$ is orthogonal to every vector. This is a standard convention.

3.2 Linear combinations and span of a set

Definition 3.4. Fix a finite set of vectors $X = \{\mathbf{x}_1, .., \mathbf{x}_m\} \subset \mathbb{R}^n$. We say that $\mathbf{x} \in \mathbb{R}^n$ can be expressed as a **linear combination** of vectors in X if there exist $\lambda_1, ..., \lambda_m \in \mathbb{R}$ such that

$$\mathbf{x} = \sum_{k=1}^{m} \lambda_k \mathbf{x}_k.$$

Definition 3.5. Fix a set of vectors $X \subset \mathbb{R}^n$. The **span of** X is given by

$$Span(X) = \left\{ \mathbf{y} \in \mathbb{R}^n \mid \mathbf{y} \text{ is a linear combination of vectors in some finite subset } \hat{X} \subset X \right\}.$$

Example 3.1. Let $\mathbf{u} = (1, 0, 0)'$, $\mathbf{v} = (0, 1, 0)'$ and let $X = \{\mathbf{u}, \mathbf{v}\}$. Then $Span(X) = \{(\lambda_1, \lambda_2, \lambda_3)' \in \mathbb{R}^3 \mid \lambda_3 = 0\}$.

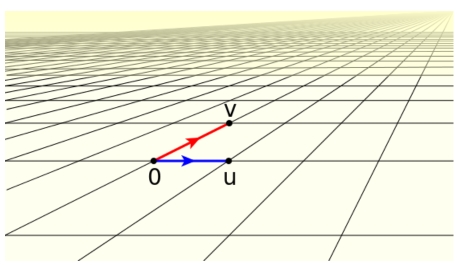


Figure 3.1 The cross-hatched plane is the span of $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$.

Proposition 3.3. *Let $X \subset \mathbb{R}^n$, and $\mathbf{x} \in \mathbb{R}^n$. $\mathbf{x}$ is orthogonal to $Span(X)$ if and only if $\mathbf{x}$ is orthogonal to X.*

Proof. Assume $\mathbf{x}$ is orthogonal to X. Let $\mathbf{y} \in Span(X)$, which means that there exist $\mathbf{x}_1, \ldots, \mathbf{x}_m \in X$, and some $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$ such that $\mathbf{y} = \sum_{k=1}^{m} \lambda_k \mathbf{x}_k$. Therefore, $\mathbf{x} \cdot \mathbf{y} = \sum_{k=1}^{m} \lambda_k (\mathbf{x} \cdot \mathbf{x}_k) = 0$, so $\mathbf{x}$ and $\mathbf{y}$ are orthogonal. Since $\mathbf{y} \in Span(X)$ was arbitrary, we conclude that $\mathbf{x}$ is orthogonal to $Span(X)$. $\square$

3.3 Linear independence

Definition 3.6. Fix a finite set of vectors $X = \{\mathbf{x}_1, \ldots, \mathbf{x}_m\} \subset \mathbb{R}^n$.

(i) Say that X is **linearly dependent** (l.d.) if there are coefficients $\lambda_1, ..., \lambda_m$, not all zero, such that $\sum_{k=1}^m \lambda_k \mathbf{x}_k = \mathbf{0}$.

(ii) Say X is **linearly independent** (l.i.) if for all $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$, $\sum_{k=1}^m \lambda_k \mathbf{x}_k = \mathbf{0}$ implies that $\lambda_j = 0$ for each $j = 1, \ldots, m$.

So, X is l.d. if and only if at least one of its elements can be expressed as a linear combination of the other elements. Equivalently, X is l.i. if and only if none of its elements can be expressed as a linear combination of the other elements. (Exercise.)

Remark 3.3. Any finite set X containing the zero vector is l.d. (Why?)

Example 3.2. The set $\left\{ \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}, \begin{pmatrix} -1 \\ -5 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} \right\}$ is l.i. (Why?)

Proposition 3.4. *Let $X \subset \mathbb{R}^n$ be a finite set. Then*

(i) If X contains an l.d. subset, then X is l.d.

(ii) If X is l.i., then any subset of X is l.i.

Proof. (i) Let $T = \{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ be a subset of $X = \{\mathbf{x}_1, \ldots, \mathbf{x}_k, \mathbf{x}_{k+1}, \ldots, \mathbf{x}_n\}$. Assume that T is l.d., so that there are $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ not all zero, such that $\lambda_1 \mathbf{x}_1 + \ldots + \lambda_k \mathbf{x}_k = 0$. Thus, $\lambda_1 \mathbf{x}_1 + \ldots + \lambda_k \mathbf{x}_k + 0\mathbf{x}_{k+1} + \ldots + 0\mathbf{x}_n = 0$, and so X is l.d.

(ii) Exercise.

$\square$

3.4 Matrices

Definition 3.7. An $n \times m$ **matrix** is a rectangular array of real numbers

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1m} \\ a_{21} & a_{22} & \ldots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nm} \end{pmatrix}_{n \times m}$$

consisting of n rows and m columns. Say A is a **square matrix** if $n = m$.

Notation 3.2. *We usually write $A = (a_{ij})_{n \times m}$ more compactly, where a_{ij} stands for the entry in the ith row and the jth column. The set of $n \times m$ matrices is denoted by $\mathbb{R}^{n \times m}$.*

Example 3.3. The matrix

$$A = \begin{pmatrix} 0 & 2 & 0 \\ 1 & 3 & 1 \end{pmatrix}$$

can be written as $A = (a_{ij}) \in \mathbb{R}^{2 \times 3}$ where $a_{ij} = i + (-1)^j$.

3.4.1 Operations with matrices

Addition and scalar multiplication for matrices are defined component-wise, as it is for vectors. That is, for $A, B \in \mathbb{R}^{n\times m}$ and $\lambda \in \mathbb{R}$, we define $A+B = (a_{ij}+b_{ij}) \in \mathbb{R}^{n\times m}$ and $\lambda A = (\lambda a_{ij}) \in \mathbb{R}^{n\times m}$.

Definition 3.8. Let $A \in \mathbb{R}^{n\times m}$. The **transpose of** A is the matrix $A' = (a_{ji}) \in \mathbb{R}^{m\times n}$.

Proposition 3.5. *Let A, B be matrices of adequate size and $\lambda \in \mathbb{R}$. Then,*

(i) $(A')' = A$.

(ii) $(A+B)' = A' + B'$.

(iii) $(\lambda A)' = \lambda A'$.

Proof. Exercise. □

Definition 3.9. Say $A \in \mathbb{R}^{n\times n}$ is **symmetric** if $A' = A$.

Example 3.4. Let $A \in \mathbb{R}^{n\times n}$. Since $(A+A')' = A' + A = A + A'$, then $A + A'$ is symmetric.

Definition 3.10. Let $A \in \mathbb{R}^{n\times m}$ and $B \in \mathbb{R}^{m\times p}$. The **matrix product** $AB = (c_{ij})$ is an $n \times p$ matrix whose ijth entry is given by $c_{ij} = \sum_{k=1}^{m} a_{ik}b_{kj}$.

Remark 3.4. Let $A \in \mathbb{R}^{n\times m}$ be a matrix. Write $A_i \in \mathbb{R}^m$ for the ith row of A, and $A^j \in \mathbb{R}^n$ for the jth column of A. Then $AB = (c_{ij})$ satisfies $c_{ij} = A_i \cdot B^j$.

Example 3.5. Consider the matrices

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}_{2\times 3}, \qquad B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}_{3\times 2}$$

Then

$$AB = \begin{pmatrix} (1\cdot 1 + 2\cdot 3 + 3\cdot 5) & (1\cdot 2 + 2\cdot 4 + 3\cdot 6) \\ (4\cdot 1 + 5\cdot 3 + 6\cdot 5) & (4\cdot 2 + 5\cdot 4 + 6\cdot 6) \end{pmatrix} = \begin{pmatrix} 22 & 28 \\ 49 & 64 \end{pmatrix}_{2\times 2}$$

Remark 3.5. The product is defined also for matrices and vectors. If $A \in \mathbb{R}^{n\times m}$ and $\mathbf{x} \in \mathbb{R}^m$, then $A\mathbf{x}$ is carried out as a matrix product considering $\mathbf{x}$ as an $m \times 1$ matrix. In other words, $A\mathbf{x} = (c_i) \in \mathbb{R}^n$ is the vector whose ith entry is given by $c_i = \sum_{k=1}^{m} a_{ik}x_k = A_i \cdot \mathbf{x}$.

Example 3.6. Consider the matrix and vector

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}_{2\times 3}, \qquad \mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \in \mathbb{R}^3$$

Then

$$A\mathbf{x} = \begin{pmatrix} 1 \cdot 1 + 2 \cdot 2 + 3 \cdot 3 \\ 4 \cdot 1 + 5 \cdot 2 + 6 \cdot 3 \end{pmatrix} = \begin{pmatrix} 14 \\ 32 \end{pmatrix} \in \mathbb{R}^2$$

Definition 3.11. Say $A \in \mathbb{R}^{n\times n}$ is **orthogonal** if $AA' = A'A = I_n$.

Definition 3.12. The **identity matrix**, denoted $I_n = (a_{ij}) \in \mathbb{R}^{n\times n}$, is the matrix such that $a_{ij} = 1$ if $i = j$, and $a_{ij} = 0$ else. Note that I_n is symmetric and orthogonal.

Proposition 3.6. *Let $A \in \mathbb{R}^{n\times n}$. If A is orthogonal, then*

(i) each of its rows (columns) has norm 1. This is, $\|A_i\| = 1$ ($\|A^i\| = 1$) for each $i = 1, \ldots, n$; and

(ii) each of its rows (columns) is orthogonal to each other. This is, $A_i \cdot A_j = 0$ ($A^i \cdot A^j = 0$) for $i \neq j$.

Proof. Exercise. □

Proposition 3.7. *(Properties of matrix multiplication)*

Let A, B, C be matrices of adequate size and $\lambda \in \mathbb{R}$. Then,

(i) The multiplication of matrices is, in general, not commutative. This is, $AB \neq BA$.

(ii) The multiplication of matrices satisfies associativity, that is, $(AB)C = A(BC)$.

(iii) The multiplication of matrices satisfies distributivity over addition, that is, $A(B + C) = AB + AC$ and $(A + B)C = AC + BC$.

(iv) $\lambda(AB) = (\lambda A)B = A(\lambda B)$.

(v) $A_{m\times n} I_n = A$ and $I_m A_{m\times n} = A$.

(vi) $(AB)' = B'A'$.

Proof. Exercise. □

Remark 3.6. Note that $CI_n = I_n C = C$ for any square matrix $C \in \mathbb{R}^{n\times n}$.

Example 3.7. Let $A \in \mathbb{R}^{n\times m}$. The matrices $A'A$ and AA' are symmetric. (Why?)

Notation 3.3. *Let $A \in \mathbb{R}^{n\times n}$ be a square matrix. Define $A^2 = AA$ (A **squared**), and $A^n = A^{n-1}A$ (A **to the nth power**) for $n \geq 2$.*

3.4.2 Invertible matrices

Definition 3.13. Fix a matrix A of size $n \times n$. Say that A is **invertible** if there exists a matrix B of size $n \times n$ such that $AB = BA = I_n$.

Proposition 3.8. *Let $A, B \in \mathbb{R}^{n \times n}$ be invertible matrices, and $\lambda \in \mathbb{R} \setminus \{0\}$.*

(i) The inverse matrix is unique. Thus, we write A^{-1} for the inverse of A.

(ii) $(A^{-1})^{-1} = A$.

(iii) $(A')^{-1} = (A^{-1})'$.

(iv) $(AB)^{-1} = B^{-1}A^{-1}$.

(v) $(\lambda A)^{-1} = \frac{1}{\lambda} A^{-1}$.

Proof. (i) Let $A, B, C \in \mathbb{R}^{n \times n}$. Assume that $AB = BA = I$ and $AC = CA = I$. Then $C = C(AB) = (CA)B = B$.

(ii) Since $A^{-1}A = I$ and the inverse is unique, we conclude that $(A^{-1})^{-1} = A$.

The remaining properties are left as exercise. □

3.4.3 Trace and determinant

Associated with any square matrix $A \in \mathbb{R}^{n \times n}$, there are two scalar quantities called the determinant and the trace of A. The **trace** of $A = (a_{ij})$ is given by $Tr(A) = \sum_{k=1}^{n} a_{kk}$.

Proposition 3.9. *Let A, B be matrices of adequate size, and $\lambda \in \mathbb{R}$. Then,*

(i) $Tr(A') = Tr(A)$.

(ii) $Tr(\lambda A) = \lambda Tr(A)$.

(iii) $Tr(A + B) = Tr(A) + Tr(B)$.

(iv) $Tr(AB) = Tr(BA)$

Proof. (i) By definition.

(ii) Let $A \in \mathbb{R}^{n \times n}$. Then $Tr(\lambda A) = \sum_{k=1}^{n} \lambda a_{kk} = \lambda \sum_{k=1}^{n} a_{kk} = \lambda Tr(A)$.

(iii) Let $A, B \in \mathbb{R}^{n \times n}$. Then $Tr(A + B) = \sum_{k=1}^{n} (a_{kk} + b_{kk}) = \sum_{k=1}^{n} a_{kk} + \sum_{k=1}^{n} b_{kk} = Tr(A) + Tr(B)$.

(iv) Let $A \in \mathbb{R}^{n \times m}$ and $B \in \mathbb{R}^{m \times n}$. Then $Tr(AB) = \sum_{k=1}^{n} (A_k \cdot B^k) = \sum_{k=1}^{n} \sum_{i=1}^{m} a_{ki} b_{ik} = \sum_{i=1}^{m} \sum_{k=1}^{n} b_{ik} a_{ki} = \sum_{i=1}^{m} (B_i \cdot A^i) = Tr(BA)$.

□

We use a recursive formula to define the **determinant** of a square matrix A, denoted $\det(A)$ or $|A|$. For the 1×1 matrix, $A = (a_{11})$, define $\det(A) = a_{11}$.

Now, given $A \in \mathbb{R}^{n\times n}$, to each of its entries a_{ij} we associate an $(n-1)\times(n-1)$ submatrix, known as the **minor** ij, and denoted M_{ij}. This submatrix is contructed from A by removing its ith row and jth column.

Example 3.8. For $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, we have $M_{11} = (a_{22})$, $M_{12} = (a_{21})$, $M_{21} = (a_{12})$, and $M_{22} = (a_{11})$.

In addition, there is a scalar associated to each of the entries of A, known as the **cofactor** ij, and defined by $c_{ij} = (-1)^{i+j}\det(M_{ij})$. Finally, the **determinant** of A is defined as

$$\det(A) = \sum_{j=1}^{n} a_{ij}c_{ij},$$

for any $i = 1, \ldots, n$.

Example 3.9. Consider $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. Then, $\det(A) = \sum_{j=1}^{2} a_{1j}c_{1j} = a_{11}a_{22} - a_{12}a_{21}$.

Remark 3.7. The determinant and the trace are defined only for square matrices.

Remark 3.8. For any $n \in \mathbb{N}$, $\det(I_n) = 1$. (Exercise)

Proposition 3.10. *Let $A, B \in \mathbb{R}^{n\times n}$, and $\lambda \in \mathbb{R}$. Then,*

(i) $\det(A) = \det(A')$

(ii) $\det(AB) = \det(A)\det(B)$

(iii) $\det(\lambda A) = \lambda^n \det(A)$.

(iv) If A is invertible, then $\det(A) \neq 0$ *and* $\det(A^{-1}) = \frac{1}{\det(A)}$.

Proof. (iv) Suppose A is invertible, then $AA^{-1} = I$. Hence, $\det(AA^{-1}) = \det(A)\det(A^{-1}) = \det(I) = 1$, and the result follows.

□

Theorem 3.1. *Let $A \in \mathbb{R}^{n\times n}$. Then, A is invertible if and only if* $\det(A) \neq 0$.

3.5 Eigenvalues

Definition 3.14. Let $A \in \mathbb{R}^{n\times n}$. Say that $\lambda \in \mathbb{R}$ is an **eigenvalue** of A if there exists $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ such that $A\mathbf{x} = \lambda\mathbf{x}$. The vector $\mathbf{x}$ is referred to as the **eigenvector** of A associated to the eigenvalue λ.

Proposition 3.11. *Let $A \in \mathbb{R}^{n\times n}$, and $\lambda \in \mathbb{R}$. Then, λ is an eigenvalue of A if and only if* $\det(A - \lambda I) = 0$.

Proof. Exercise. $\square$

Theorem 3.2. *Let $A \in \mathbb{R}^{n\times n}$. Then A is invertible if and only if zero is not one of its eigenvalues.*

Proof. A is not invertible means that $\det(A) = 0$. This is equivalent to $\det(A - \lambda I) = 0$ for $\lambda = 0$. $\square$

3.6 Linear transformations

Definition 3.15. A **linear transformation** from $\mathbb{R}^n$ into $\mathbb{R}^m$ is a function $T : \mathbb{R}^n \to \mathbb{R}^m$ that satisfies:

(i) (**Additivity**) $T(\mathbf{x}_1 + \mathbf{x}_2) = T(\mathbf{x}_1) + T(\mathbf{x}_2)$ for each $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^n$.

(ii) (**Homogeneity**) $T(\lambda \mathbf{x}) = \lambda T(\mathbf{x})$ for each $\mathbf{x} \in \mathbb{R}^n$ and each $\lambda \in \mathbb{R}$.

Intuitively, a linear transformation is a function that is compatible with the sum and scalar multiplication of each space.

Example 3.10. Let $T : \mathbb{R} \to \mathbb{R}$ be a linear transformation. Note that for every $x \in \mathbb{R}$, we have that $T(x) = T(x1) = xT(1)$. Writing the constant $T(1) = m$, we conclude that $T(x) = mx$.

Proposition 3.12. *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then,*

(i) $T(\mathbf{0}) = \mathbf{0}$.

(ii) If $X = \{\mathbf{x}_1, \ldots, \mathbf{x}_k\} \subset \mathbb{R}^n$ is l.d. in $\mathbb{R}^n$, then $T(X)$ is l.d. in $\mathbb{R}^m$.

(iii) T is injective if and only if $T(\mathbf{x}) = \mathbf{0}$ implies $\mathbf{x} = \mathbf{0}$.

(iv) T is onto if and only if Span$(\{T(\mathbf{e}_i) \mid i = 1, \ldots, n\}) = \mathbb{R}^m$.

(v) T is continuous.

Proof. (i) Note that $T(\mathbf{0}) = T(\mathbf{0} + \mathbf{0}) = T(\mathbf{0}) + T(\mathbf{0})$. Thus, $T(\mathbf{0}) = \mathbf{0}$.

(ii) Since X is l.d., there are $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ not all zero, such that $\sum_{i=1}^k \lambda_i \mathbf{x}_i = \mathbf{0}$. Hence, $T(\sum_{i=1}^k \lambda_i \mathbf{x}_i) = \sum_{i=1}^k \lambda_i T(\mathbf{x}_i) = \mathbf{0}$. So, $T(X)$ is l.d.

(iii) Assume T is injective. Since $T(\mathbf{0}) = \mathbf{0}$, then $T(\mathbf{x}) = \mathbf{0}$ implies $\mathbf{x} = \mathbf{0}$.

Suppose now that $T(\mathbf{x}) = \mathbf{0}$ implies $\mathbf{x} = \mathbf{0}$. Let $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^n$, if $T(\mathbf{x}_1) = T(\mathbf{x}_2)$ then $\mathbf{0} = T(\mathbf{x}_1) - T(\mathbf{x}_2) = T(\mathbf{x}_1 - \mathbf{x}_2)$. Finally, $\mathbf{x}_1 = \mathbf{x}_2$.

(iv) Assume that T is onto, and let $\mathbf{y} \in \mathbb{R}^m$. Then, there exists $\mathbf{x} = \sum_{i=1}^n x_i\mathbf{e}_i \in \mathbb{R}^n$ such that $\mathbf{y} = T(\mathbf{x}) = T(\sum_{i=1}^n x_i\mathbf{e}_i) = \sum_{i=1}^n x_i T(\mathbf{e}_i)$. Thus, $y \in Span(\{T(\mathbf{e}_i \mid i = 1, \ldots, n)\})$.

The converse is similar.

□

Example 3.11. Let $A \in \mathbb{R}^{m\times n}$. The function $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by $T(\mathbf{x}) = A\mathbf{x}$ is a linear transformation.

Theorem 3.3. *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then, there exists a unique matrix $A \in \mathbb{R}^{m\times n}$ such that $T(\mathbf{x}) = A\mathbf{x}$ for each $\mathbf{x} \in \mathbb{R}^n$. The matrix A is called the **representation matrix** of T.*

Proof. We only show existence, leaving uniqueness as an exercise. Define $A = (T(\mathbf{e}_1), \cdots, T(\mathbf{e}_n))$ that is, A is the matrix whose jth column is $T(\mathbf{e}_j) \in \mathbb{R}^m$. So, $A\mathbf{e}_i = T(\mathbf{e}_i)$. Fix $\mathbf{x} \in \mathbb{R}^n$ and notice that

$$\begin{aligned} A\mathbf{x} &= A\left(\sum_{i=1}^n x_i\mathbf{e}_i\right) \\ &= \sum_{i=1}^n x_i(A\mathbf{e}_i) \\ &= \sum_{i=1}^n x_i T(\mathbf{e}_i) \\ &= T\left(\sum_{i=1}^n x_i\mathbf{e}_i\right) \\ &= T(\mathbf{x}), \end{aligned}$$

as desired. □

Remark 3.9. Notice that the proof shows how to construct the matrix of any linear transformation. The matrix's columns are given by the image of $\mathbf{e}_1, \ldots, \mathbf{e}_n$ under T.

Proposition 3.13. *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let A be its matrix representation. Then,*

(i) T is injective if and only if the linear system $A\mathbf{x} = \mathbf{0}$ has a unique solution.

(ii) T is onto if and only if $Span(\{A^i \mid i = 1, \ldots, n\}) = \mathbb{R}^m$.

Proof. (i) T is injective if and only if $T(\mathbf{x}) = \mathbf{0}$ implies $\mathbf{x} = 0$. This is to say, $A\mathbf{x} = 0$ implies $\mathbf{x} = 0$, so that the system has a unique solution.

(ii) Immediate since $T(\mathbf{e}_i) = A^i$.

□

Theorem 3.4. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Then T is bijective if and only if A is invertible.*

3.7 Quadratic forms

Definition 3.16. A **quadratic form** $q : \mathbb{R}^n \to \mathbb{R}$ is a function of the form $q(\mathbf{x}) = \sum_{i=1}^n \sum_{j=1}^n a_{ij} x_i x_j$, where $a_{ij} \in \mathbb{R}$.

Example 3.12. The function $q : \mathbb{R} \to \mathbb{R}$, $x \mapsto ax^2$ is a quadratic form. Note that

- If $a > 0$, then $q(x) > 0$ for all $x \neq 0$. If $a \geq 0$, $q(x) \geq 0$ for all x.
- If $a < 0$, then $q(x) < 0$ for all $x \neq 0$. If $a \leq 0$, $q(x) \leq 0$ for all x.

So, univariate quadratic forms are *sign-regular*, meaning that for any univariate quadratic form q we cannot find x, y such that $q(x) > 0$ and $q(y) < 0$.

Example 3.13. Multivariate quadratic forms need not exhibit sign regularity. Consider $q(x_1, x_2) = x_1 x_2$. Observe that $q(1, 1) > 0$ and $q(-1, 1) < 0$.

Proposition 3.14. *Let $q : \mathbb{R}^n \to \mathbb{R}$ be a quadratic form. There exists a unique symmetric matrix $A \in \mathbb{R}^{n \times n}$ such that $q(\mathbf{x}) = \mathbf{x}' A \mathbf{x}$, for each $\mathbf{x} \in \mathbb{R}^n$. The matrix A is referred to as the **matrix representation** of q.*

Proof. We prove existence, leaving uniqueness as an exercise. Let $q : \mathbb{R}^n \to \mathbb{R}$ be a quadratic form, so it can be written as $q(\mathbf{x}) = \sum_{i=1}^n \sum_{j=1}^n \hat{a}_{ij} x_i x_j$. Consider the matrix $\hat{A} = (\hat{a}_{ij}) \in \mathbb{R}^{n \times n}$. Then, for any $\mathbf{x} \in \mathbb{R}^n$, $\hat{A}\mathbf{x} = (\hat{A}_i \cdot \mathbf{x})_i = (\sum_{j=1}^n \hat{a}_{ij} x_j)_i \in \mathbb{R}^n$. Hence,

$$\begin{aligned}
\mathbf{x}' \hat{A} \mathbf{x} &= \mathbf{x}' \left(\sum_{j=1}^n \hat{a}_{ij} x_j \right)_i \\
&= \sum_{i=1}^n x_i \left(\sum_{j=1}^n \hat{a}_{ij} x_j \right) \\
&= \sum_{i=1}^n \sum_{j=1}^n \hat{a}_{ij} x_i x_j,
\end{aligned}$$

so $q(\mathbf{x}) = \mathbf{x}' \hat{A} \mathbf{x}$ for any $\mathbf{x} \in \mathbb{R}^n$.

Now, let $A = \frac{1}{2}(\hat{A} + \hat{A}')$. Note that A is symmetric and that $\mathbf{x}' A \mathbf{x} = \frac{1}{2}(\mathbf{x}' \hat{A} \mathbf{x} + \mathbf{x}' \hat{A}' \mathbf{x}) = \mathbf{x}' \hat{A} \mathbf{x} = q(\mathbf{x})$, where the second equality holds since $\mathbf{x}' \hat{A}' \mathbf{x} = (\mathbf{x}' \hat{A}' \mathbf{x})' = \mathbf{x}' \hat{A} \mathbf{x}$. □

Remark 3.10. Proposition 3.14 establishes a one-to-one correspondence between symmetric matrices and quadratic forms.

Definition 3.17. Let $q(\mathbf{x}) = \mathbf{x}'A\mathbf{x}$ be a quadratic form. Say q (or A) is

(i) **Positive definite** (PD) if $q(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{0}$.

(ii) **Positive semidefinite** (PSD) if $q(\mathbf{x}) \geq 0$ for all $\mathbf{x}$.

(iii) **Negative definite** (ND) if $q(\mathbf{x}) < 0$ for all $\mathbf{x} \neq \mathbf{0}$.

(iv) **Negative semidefinite** (NSD) if $q(\mathbf{x}) \leq 0$ for all $\mathbf{x}$.

(v) **Indefinite** if there are $\mathbf{x}, \mathbf{y}$ such that $q(\mathbf{x}) > 0$ and $q(\mathbf{y}) < 0$.

Remark 3.11. Note that any positive (negative) definite quadratic form is positive (negative) semidefinite.

Example 3.14. The quadratic form $q(x_1, x_2) = (x_1 - x_2)^2$ is PSD but not PD.

Definition 3.18. Let $\Gamma \in \mathbb{R}^{n\times n}$. Say Γ is **lower triangular** if $\gamma_{ij} = 0$ for $i < j$.

Theorem 3.5. *(Cholesky decomposition)*

*Let $\Omega \in \mathbb{R}^{n\times n}$ be symmetric. If Ω is PD, there exists a unique invertible lower triangular matrix $\Gamma \in \mathbb{R}^{n\times n}$ such that $\Omega = \Gamma\Gamma'$. The matrix Γ is called the **Cholesky decomposition** of Ω.*

Remark 3.12. The Cholesky decomposition is a result frequently invoked in econometrics. This is so because every variance-covariance matrix is PD.

Proposition 3.15. *Let $q(\mathbf{x}) = \mathbf{x}'A\mathbf{x}$ be a quadratic form. Then*

(i) q (or A) is PD (ND) if and only if all eigenvalues of A are positive (negative).

(ii) q (or A) is PSD (NSD) if and only if all eigenvalues of A are non-negative (non-positive).

Proof. (Only sufficiency)

(i) Assume q is PD, and let $\lambda \in \mathbb{R}$ be one of its eigenvalues. Then there exists $\mathbf{x} \neq \mathbf{0}$ such that $A\mathbf{x} = \lambda\mathbf{x}$. Hence, $0 < \mathbf{x}'A\mathbf{x} = \mathbf{x}'(\lambda\mathbf{x}) = \lambda(\mathbf{x}'\mathbf{x})$. So, $\lambda > 0$ because $\mathbf{x}'\mathbf{x} > 0$.

(ii) Similar.

□

Corollary 3.1. *Let $A \in \mathbb{R}^{n\times n}$ be symmetric. If A is PD (ND), then A is invertible and A^{-1} is PD (ND) as well.*

Proof. The matrix A is PD if and only if all of its eigenvalues λ are positive. Hence, all eigenvalues $\frac{1}{\lambda}$ of A^{-1} are positive, and so A^{-1} is PD. □

Corollary 3.2. *If $A \in \mathbb{R}^{n\times n}$ is PSD (NSD) but not PD (ND), then A is not invertible.*

Definition 3.19. Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ be a square matrix. The **leading principal minor of order** r, $1 \leq r \leq n$; is the determinant

$$M_r = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1r} \\ a_{21} & a_{22} & \dots & a_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r1} & a_{r2} & \dots & a_{rr} \end{vmatrix}_{r\times r}.$$

A **principal minor of order** r, $1 \leq r \leq n$; is a determinant of the form

$$\begin{vmatrix} a_{ii} & a_{ij} & \dots & a_{ik} \\ a_{ji} & a_{jj} & \dots & a_{jk} \\ \vdots & \vdots & \ddots & \vdots \\ a_{ki} & a_{kj} & \dots & a_{kk} \end{vmatrix}_{r\times r}.$$

where $i < j < \ldots < k$. The set of all principal minors of order r is denoted by P_r.

Remark 3.13. A principal minor of order r is the determinant of a submatrix constructed from A by selecting r rows and the corresponding columns.

Theorem 3.6. *Let $A \in \mathbb{R}^{n\times n}$ be symmetric. Then*

(i) A is PD if and only if $M_r > 0$ for all $r = 1, \ldots, n$.

(ii) A is PSD if and only if $P_r \geq 0$ for all $r = 1, \ldots, n$.

(iii) A is ND if and only if $(-1)^r M_r > 0$ for all $r = 1, \ldots, n$.

(iv) A is NSD if and only if $(-1)^r P_r \geq 0$ for all $r = 1, \ldots, n$.

Remark 3.14. In Theorem 3.6, $P_r \geq 0$ means that all elements of P_r are nonnegative.

Remark 3.15. A square matrix $A \in \mathbb{R}^{n\times n}$ does not satisfy any conditions in Theorem 3.6 if and only if it is indefinite.

3.8 Matrix differentiation

This subsection summarizes some of the most used matrix derivatives in econometrics. It is mainly a notational effort.

Definition 3.20. Let $\mathbf{y} \in \mathbb{R}^m$ and $\mathbf{x} \in \mathbb{R}^n$. If $\mathbf{y} = f(\mathbf{x})$ for some function $f : \mathbb{R}^n \to \mathbb{R}^m$, we define **the partial derivative of the vector y with respect to the vector x** by

$$\frac{\partial \mathbf{y}}{\partial \mathbf{x}} = \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial y_m}{\partial x_1} & \frac{\partial y_m}{\partial x_2} & \cdots & \frac{\partial y_m}{\partial x_n} \end{pmatrix} \in \mathbb{R}^{m \times n}$$

Example 3.15. If $y \in \mathbb{R}$, $\mathbf{x} \in \mathbb{R}^n$, and $y = f(\mathbf{x})$ for some $f : \mathbb{R}^n \to \mathbb{R}$, then

$$\frac{\partial y}{\partial \mathbf{x}} = \begin{pmatrix} \frac{\partial y}{\partial x_1} & \frac{\partial y}{\partial x_2} & \cdots & \frac{\partial y}{\partial x_n} \end{pmatrix} = \nabla f(\mathbf{x})'$$

Example 3.16. Let $\mathbf{y}, \mathbf{x} \in \mathbb{R}^n$. Then $\lambda = \mathbf{y}'\mathbf{x} = \sum_{j=1}^n y_j x_j \in \mathbb{R}$, and so $\frac{\partial \lambda}{\partial x_i} = y_i$. In conclusion, $\frac{\partial \lambda}{\partial \mathbf{x}} = \mathbf{y}'$.

A similar argument applies to show that $\frac{\partial \lambda}{\partial \mathbf{y}} = \mathbf{x}'$

Example 3.17. Let $\mathbf{y} \in \mathbb{R}^m$ and $\mathbf{x} \in \mathbb{R}^n$. If $\mathbf{y} = A\mathbf{x}$ for some constant matrix $A \in \mathbb{R}^{m\times n}$, then $\mathbf{y} = A\mathbf{x} = (A_i \cdot \mathbf{x})_i = (\sum_{j=1}^n a_{ij} x_j)_i$. Thus, $\frac{\partial y_i}{\partial x_j} = a_{ij}$, which implies that $\frac{\partial \mathbf{y}}{\partial \mathbf{x}} = A$.

Example 3.18. Let $\mathbf{y} \in \mathbb{R}^m$, $\mathbf{x} \in \mathbb{R}^n$, and $\mathbf{z} \in \mathbb{R}^p$. Assume that $\mathbf{x} = f(\mathbf{z})$ for some $f : \mathbb{R}^p \to \mathbb{R}^n$, and that $\mathbf{y} = A\mathbf{x}$ for some constant $A \in \mathbb{R}^{m\times n}$. Then, $\mathbf{y} = A\mathbf{x} = (A_i \cdot \mathbf{x})_i = (\sum_{k=1}^n a_{ik} x_k)_i$. Consequently, $\frac{\partial y_i}{\partial z_j} = \sum_{k=1}^n a_{ik} \frac{\partial x_k}{\partial z_j} = A_i \cdot (\frac{\partial x}{\partial z})^j$. Thus, $\frac{\partial \mathbf{y}}{\partial \mathbf{z}} = \frac{\partial \mathbf{y}}{\partial \mathbf{x}} \frac{\partial \mathbf{x}}{\partial \mathbf{z}} = A \frac{\partial \mathbf{x}}{\partial \mathbf{z}}$.

Example 3.19. Let $\lambda = \mathbf{y}' A \mathbf{x} \in \mathbb{R}$, where $\mathbf{y} \in \mathbb{R}^m$, $\mathbf{x} \in \mathbb{R}^n$, and $A \in \mathbb{R}^{m\times n}$ is constant. Then,

- Write $\mathbf{w}' = \mathbf{y}'A \in \mathbb{R}^n$. Hence, $\lambda = \mathbf{w}'\mathbf{x}$. As a result, $\frac{\partial \lambda}{\partial \mathbf{x}} = \mathbf{w}' = \mathbf{y}'A$.
- Write $\mathbf{z} = A\mathbf{x} \in \mathbb{R}^m$. Hence, $\lambda = \mathbf{y}'\mathbf{z}$. As a result, $\frac{\partial \lambda}{\partial \mathbf{y}} = \mathbf{z}' = \mathbf{x}'A'$.

Example 3.20. Let $\lambda = \mathbf{x}'A\mathbf{x}$ for $\mathbf{x} \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$ constant. Since $\lambda = \sum_{j=1}^n \sum_{k=1}^n a_{kj} x_k x_j$, then $\frac{\partial \lambda}{\partial x_i} = \sum_{j=1}^n a_{ij} x_j + \sum_{k=1}^n a_{ki} x_k = (A_i \cdot \mathbf{x})_i + (A^i \cdot \mathbf{x})_i$. Finally, $\frac{\partial \lambda}{\partial \mathbf{x}} = \mathbf{x}'A' + \mathbf{x}'A = \mathbf{x}'(A + A')$.

If in addition A is symmetric, $\frac{\partial \lambda}{\partial \mathbf{x}} = 2\mathbf{x}'A$

Example 3.21. Let $\lambda = \mathbf{x}'A\mathbf{x}$ for $\mathbf{x} \in \mathbb{R}^n$, and $A \in \mathbb{R}^{n\times n}$ constant and symmetric. Assume $\mathbf{x}$ is a function of $\mathbf{z} \in \mathbb{R}^m$. Then, $\frac{\partial \lambda}{\partial \mathbf{z}} = 2\mathbf{x}'A\frac{\partial \mathbf{x}}{\partial \mathbf{z}}$. (Why?)

3.9 Exercises

(i) Prove Proposition 3.1.

(ii) Prove Proposition 3.2.

(iii) Let $X \subset \mathbb{R}^n$ be a finite set. Show that X is l.d. if and only if one of its elements is a linear combination of the remaining elements of the set.

(iv) Let $X \subset \mathbb{R}^n$ be a finite set. Prove that if $0 \notin X$ and the elements of X are orthogonal to each other, then X is l.i.

(v) Complete the proof of Proposition 3.4.

(vi) Prove Proposition 3.5.

(vii) Prove Proposition 3.6.

(viii) Prove Proposition 3.7.

(ix) Let $A \in \mathbb{R}^{n\times n}$. Say A is **idempotent** if $A^n = A$ for each $n \in \mathbb{N}$.

Prove that A is idempotent if and only if $A^2 = A$.

(x) Complete the proof of Proposition 3.8.

(xi) Show that if A is a square matrix and B is an invertible square matrix of the same size, then $Tr(B^{-1}AB) = Tr(A)$.

(xii) Show that $\det(I_n) = 1$ for each $n \in \mathbb{N}$.

(xiii) Show that any scalar multiple of an eigenvector is an eigenvector. What is the associated eigenvalue?

(xiv) Prove Proposition 3.11.

(xv) Let $A \in \mathbb{R}^{n\times n}$ be an invertible matrix. Prove that $\lambda \in \mathbb{R}$ is an eigenvalue of A if and only if $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.

(xvi) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Under which condition T maps l.i. sets into l.i. sets? Why?

(xvii) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Show that the matrix representation of T is unique. That is, if $T(\mathbf{x}) = A\mathbf{x}$ and $T(\mathbf{x}) = B\mathbf{x}$ for each $\mathbf{x} \in \mathbb{R}^n$ then $A = B$.

(xviii) Let $T_1, T_2 : \mathbb{R}^n \to \mathbb{R}^m$ be linear transformations, and $\lambda \in \mathbb{R}$. Show that

(a) $T_1 + T_2$ is a linear transformation. Find its matrix representation.

(b) λT_1 is a linear transformation. Find its matrix representation.

(xix) Let $T_1 : \mathbb{R}^n \to \mathbb{R}^m$ and $T_2 : \mathbb{R}^m \to \mathbb{R}^p$ be linear transformations. Show that $T_2 \circ T_1$ is a linear transformation and find its matrix representation.

(xx) Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a bijective linear transformation. Show that T^{-1} is a linear transformation and find its matrix representation.

(xxi) Prove that the matrix representation of any quadratic form is unique.

(xxii) Let $A \in \mathbb{R}^{n\times m}$. Show that $A'A$ and AA' are symmetric and PSD.

(xxiii) Let $A, B \in \mathbb{R}^{n\times n}$ be symmetric, and $\lambda_1, \lambda_2 \geq 0$. Show that $\lambda_1 A + \lambda_2 B$ is PD (ND) provided that A and B are PD (ND).

(xxiv) Let $A, B \in \mathbb{R}^{n\times n}$ be symmetric, and $\lambda_1, \lambda_2 \geq 0$. Show that $\lambda_1 A + \lambda_2 B$ is PSD (NSD) provided that A and B are PSD (NSD).

(xxv) Let $A \in \mathbb{R}^{n\times n}$ be symmetric. Show that $a_{ii} > 0$ (< 0) for all $i = 1, \ldots, n$, on the condition that A is PD (ND).

(xxvi) Let $A \in \mathbb{R}^{n\times n}$ be symmetric. Show that $a_{ii} \geq 0$ (≤ 0) for all $i = 1, \ldots, n$, on the condition that A is PSD (NSD).

(xxvii) Let $\lambda = \mathbf{x}'A\mathbf{x}$ for $\mathbf{x} \in \mathbb{R}^n$, and $A \in \mathbb{R}^{n\times n}$ constant and symmetric. Assume $\mathbf{x}$ is a function of $\mathbf{z} \in \mathbb{R}^m$. Show that $\frac{\partial\lambda}{\partial\mathbf{z}} = 2\mathbf{x}'A\frac{\partial\mathbf{x}}{\partial\mathbf{z}}$.

4 Convex Analysis

4.1 Convex sets

Definition 4.1. Let $X \subset \mathbb{R}^n$. Say X is a **convex set** if for all $x_0, x_1 \in X$, and each $t \in [0,1]$, it holds that $x_t = tx_1 + (1-t)x_0 \in X$.

Remark 4.1. For a given $x_1, x_0 \in \mathbb{R}^n$, the set $\{x_t \mid t \in [0,1]\}$ can be understood as the segment joining x_0 and x_1.

Example 4.1. In $\mathbb{R}$, a set is convex if and only if it is an interval. Indeed, assume $I \subset \mathbb{R}$ is an interval. Let $x_0, x_1 \in I$, and $t \in [0,1]$. We can assume, without loss of generality, that $x_0 < x_1$. So, $x_0 \leq x_t = tx_1 + (1-t)x_0 \leq x_1$. Thus, $x_t \in I$ and I is convex.

Now, to prove the converse, we proceed by contrapositive. Assume that $I \subset \mathbb{R}$ is not an interval. Hence, there are $x_0, x_1, x \in \mathbb{R}$ such that $x_0 < x < x_1$ and $x_0, x_1 \in I$ but $x \notin I$. Since the function $f : [0,1] \to [x_0, x_1]$ defined by $f(t) = x_t$ is continuous (Why?), then Theorem 2.2 ensures that there exists $t^* \in [0,1]$ such that $f(t^*) = x$. Thus, $x_{t^*} \notin I$, so I is not convex.

Example 4.2. Fix $\epsilon > 0$. The closed ball $B[0,\epsilon] = \{x \in \mathbb{R}^n \mid \|x\| \leq \epsilon\}$ is a convex set.

Let $x_1, x_0 \in B[0,\epsilon]$ and $t \in [0,1]$. Then,

$$\begin{aligned}
\|x_t\| &= \|tx_1 + (1-t)x_0\| \\
&\leq \|tx_1\| + \|(1-t)x_0\| \\
&= t\|x_1\| + (1-t)\|x_0\| \\
&\leq t\epsilon + (1-t)\epsilon \\
&= \epsilon.
\end{aligned}$$

Thus, $x_t \in B[0,\epsilon]$.

Proposition 4.1. *Let $X_1, X_2 \subset \mathbb{R}$ be convex sets and $\alpha, \beta \in \mathbb{R}$. Then,*

(i) $X_1 \cap X_2$ is a convex set.

(ii) $\alpha X_1 + \beta X_2 = \{\alpha x_1 + \beta x_2 \mid x_1 \in X_1 \wedge x_2 \in X_2\}$ is a convex set.

(iii) $X_1 \times X_2$ is a convex set.

(iv) If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then $T(X_1)$ is a convex set in $\mathbb{R}^m$.

Proof. (i) Let $x_1, x_0 \in X_1 \cap X_2$ and $t \in [0,1]$. Then $x_1, x_2 \in X_1$ and $x_1, x_2 \in X_2$. Since X_1, X_2 are convex sets, then $x_t \in X_1$ and $x_t \in X_2$. Thus, $x_t \in X_1 \cap X_2$.

(ii) Exercise.

(iii) Exercise.

(iv) Let $y_0, y_1 \in T(X_1)$ and $t \in [0,1]$. Then, there are $x_0, x_1 \in X_1$ such that $T(x_0) = y_0$ and $T(x_1) = y_1$. Now,

$$\begin{aligned} y_t &= ty_1 + (1-t)y_0 \\ &= tT(x_0) + (1-t)T(x_1) \\ &= T(x_t), \text{ since } T \text{ is linear.} \end{aligned}$$

Finally, since X_1 is convex, $x_t \in X_1$. Consequently, $y_t = T(x_t) \in T(X_1)$.

$\square$

Remark 4.2. Note that all but item (iv) in Proposition 4.1 can be extended using mathematical induction. Moreover, it is also true that arbitrary intersection of convex sets is convex (Why?).

Example 4.3. Fix $y \in \mathbb{R}^n$ and $\epsilon > 0$. The closed ball $B[y, \epsilon] = \{x \in \mathbb{R}^n \mid \|x - y\| \leq \epsilon\}$ is convex. Indeed, note that the sets $B[0, \epsilon]$ and $\{y\}$ are convex. Hence, $\{y\} + B[0, \epsilon] = \{y + x \mid \|x\| \leq \epsilon\}$ is a convex set.

Now, let us check that $\{y\} + B[0, \epsilon] = B[y, \epsilon]$, so the proof would be completed. Let $y + x \in \{y\} + B[0, \epsilon]$, then $\|(y + x) - y\| = \|x\| \leq \epsilon$, so $y + x \in B[y, \epsilon]$. Now, let $x \in B[y, \epsilon]$, so $\|x - y\| \leq \epsilon$. As a result, $\|y - x\| \leq \epsilon$, i.e., $y - x \in B[0, \epsilon]$. Thus, $x = y + (y - x) \in \{y\} + B[0, \epsilon]$.

Example 4.4. Every open ball is a convex set.

Remark 4.3. Item (iv) in Proposition 4.1 states that linear transformations preserve convexity. Recall that a linear transformation can rotate, move around, contract, or stretch the set under consideration.

Thus, by the previous example, we conclude that any rotated, contracted, or stretched ball is convex. For instance, any elliptic region is a convex set.

Remark 4.4. Convexity is, in general, not preserved by continuous mappings. (Why?)

Definition 4.2. A vector $x \in \mathbb{R}^n$ is said to be a **convex combination** of $x_0, x_1, \ldots, x_q \in \mathbb{R}^n$ if

$$x = \sum_{k=0}^{q} \lambda_k x_k,$$

for some $\lambda_0, \ldots, \lambda_q \geq 0$ satisfying $\sum_{k=0}^{q} \lambda_k = 1$.

Definition 4.3. Let $X \subset \mathbb{R}^n$ be nonempty. The **convex hull** of X, denoted $Conv(X)$, is the set of all convex combinations of vectors in X.

Proposition 4.2. *Let $X, Y \subset \mathbb{R}^n$. Then,* *(i) $Conv(X)$ is convex.*

(ii) X is convex if and only if $X = Conv(X)$.

(iii) $X \subset Y$ implies $Conv(X) \subset Conv(Y)$.

(iv) $Conv(X) = \bigcap\{C \subset \mathbb{R}^n \mid X \subset C \wedge C$ is convex$\}$. In words, $Conv(X)$ is the intersection of all convex sets containing X.

Proof. (i) Let $x_0 = \sum_{k=0}^{q} \lambda_k x_0^k \in Conv(X)$, $x_1 = \sum_{j=0}^{m} \lambda_j x_1^j \in Conv(X)$, and $t \in [0,1]$. Then $x_t = t\left(\sum_{j=0}^{m} \lambda_j x_1^j\right) + (1-t)\left(\sum_{k=0}^{q} \lambda_k x_0^k\right) = \sum_{j=0}^{m}(t\lambda_j)x_1^j + \sum_{k=0}^{q}((1-t)\lambda_k)x_0^k$. Since $\sum_{j=0}^{m}(t\lambda_j) + \sum_{k=0}^{q}((1-t)\lambda_k) = t\sum_{j=0}^{m}(\lambda_j) + (1-t)\sum_{k=0}^{q}(\lambda_k) = t + (1-t) = 1$, we conclude that $x_t \in Conv(X)$.

(ii) Suppose X is convex. We want to prove that for any $x_0, x_1, \ldots, x_q \in X$ and $\lambda_0, \lambda_1, \ldots, \lambda_q \geq 0$ with $\sum_{k=0}^{q} \lambda_k = 1$, it holds that $x = \sum_{k=0}^{q} \lambda_k x_k \in X$. We proceed by induction on q.

For $q = 1$, the property of interest holds by definition of convex set. Now, assume it holds for $q \geq 1$. Let $x_0, x_1, \ldots, x_q, x_{q+1} \in X$ and $\lambda_0, \lambda_1, \ldots, \lambda_q, \lambda_{q+1} \geq 0$ with $\sum_{k=0}^{q+1} \lambda_k = 1$. Define $\lambda'_j = \frac{\lambda_j}{\sum_{k=0}^{q} \lambda_k} = \frac{\lambda_j}{1-\lambda_{q+1}}$ for $j = 0, 1, \ldots, q$; and observe that $\sum_{j=0}^{q} \lambda'_j = 1$. Hence, $\sum_{j=0}^{q} \lambda'_j x_j \in X$ by the inductive hypothesis. Since X is convex, then $(1-\lambda_{q+1})\sum_{j=0}^{q} \lambda'_j x_j + \lambda_{q+1}x_{q+1} = \sum_{j=0}^{q+1} \lambda_j x_j \in X$, which is what we wanted to show.

(iii) Exercise.

(iv) Clearly, $X \subset Conv(X)$. By item (i), $Conv(X)$ is a convex set so that $\bigcap\{C \subset \mathbb{R}^n \mid X \subset C \wedge C$ is convex$\} \subset Conv(X)$. Now, let $C \subset \mathbb{R}^n$ be a convex set such that $X \subset C$. Hence, item (iii) and (ii) imply that $Conv(X) \subset Conv(C) = C$. Since C was arbitrary, we conclude that $Conv(X) \subset \bigcap\{C \subset \mathbb{R}^n \mid X \subset C \wedge C$ is convex$\}$.

□

Corollary 4.1. *Let $X \subset \mathbb{R}^n$. Then $Conv(Conv(X)) = Conv(X)$.*

Remark 4.5. The notion of convex set is not exclusive of the euclidean space $\mathbb{R}^n$. We can define it more generally in any *vector space* over the real field. Intuitively, a vector space is a triple $(V, +, \cdot)$ where V is a set and the operations $+, \cdot$ are *well-behaved.* For instance, the set of positive definite matrices is convex, and the set of invertible matrices is not convex. (Why?)

4.2 Convex and concave functions

Fix the convex set $X \subset \mathbb{R}^n$.

Definition 4.4. Let $f : X \to \mathbb{R}$ be a function. Then,

(i) f is **convex** if $f(x_t) \leq tf(x_1) + (1-t)f(x_0)$, for each $x_0, x_1 \in X$ and $t \in [0,1]$.

(ii) f is **strictly convex** if $f(x_t) < tf(x_1) + (1-t)f(x_0)$, for each different $x_0, x_1 \in X$ and $t \in (0,1)$.

(iii) f is **concave** if $f(x_t) \geq tf(x_1) + (1-t)f(x_0)$, for each $x_0, x_1 \in X$ and $t \in [0,1]$.

(iv) f is **strictly concave** if $f(x_t) > tf(x_1) + (1-t)f(x_0)$, for each different $x_0, x_1 \in X$ and $t \in (0,1)$.

Intuitively, a function is (strictly) convex if the line segment between any two different points of the graph lies (strictly) above the graph. Similarly, a function is (strictly) concave if the line segment between any two different points of the graph lies (strictly) below the graph.

Remark 4.6. Observe that the *strictness* rules out the possibility of straight segments in the graph of the functions.

Example 4.5. Let $q(x) = x'Ax$ be a quadratic form. q is PD if and only if it is strictly convex.

Assume q is PD, and fix $x_1, x_0 \in \mathbb{R}^n$ different, and $t \in (0,1)$, then

$$\begin{aligned} q(x_t) &= (tx_1 + (1-t)x_0)'A(tx_1 + (1-t)x_0) \\ &= (x_0 + t(x_1 - x_0))'A(x_0 + t(x_1 - x_0)) \\ &= x_0'Ax_0 + 2tx_0'A(x_1 - x_0) + t^2(x_1 - x_0)'A(x_1 - x_0) \end{aligned}$$

Since $t^2 < t$ and q is PD,

$$\begin{aligned} &< x_0'Ax_0 + 2tx_0'A(x_1 - x_0) + t(x_1 - x_0)'A(x_1 - x_0) \\ &= x_0'Ax_0 - tx_0'Ax_0 + tx_1'Ax_1 \\ &= tx_1'Ax_1 + (1-t)x_0'Ax_0 \\ &= tq(x_1) + (1-t)q(x_0). \end{aligned}$$

Thus, q is strictly convex.

On the other hand, assume that q is strictly convex, and let $x \neq 0$. Then, $q\left(\frac{1}{2}x + \frac{1}{2} \cdot 0\right) < \frac{1}{2}q(x) + \frac{1}{2}q(0)$, which is equivalent to $\frac{1}{4}q(x) < \frac{1}{2}q(x)$, which in turn implies that $q(x) > 0$.

Remark 4.7. By a similar argument, we can show that a quadratic form is:

(i) PSD if and only if it is convex.

(ii) ND if and only if it is strictly concave,

(iii) NSD if and only if it is concave.

Example 4.6. The function $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto x^3$ is neither concave nor convex. (Why?)

Proposition 4.3. *Let $f : X \to \mathbb{R}$ be a function. Then, f is (strictly) convex if and only if $-f$ is (strictly) concave.*

Proof. By definition. □

Definition 4.5. Let $f : X \to \mathbb{R}$ be a function.

(i) The hypograph of f is the set $hyp(f) = \{(x, y) \in X \times \mathbb{R} \mid y \leq f(x)\}$. In words, the hypograph of f is the set of points (x, y) lying below the graph of f.

(ii) The epigraph of f is the set $epi(f) = \{(x, y) \in X \times \mathbb{R} \mid y \geq f(x)\}$. In words, $epi(f)$ is the set of points (x, y) lying above the graph of f.

Proposition 4.4. *Let $f : X \to \mathbb{R}$ be a function. Then,*

(i) f is convex if and only if $epi(f)$ is a convex set.

(ii) f is concave if and only if $hyp(f)$ is a convex set.

Proof. Exercise. □

Proposition 4.5. *Let $X \subset \mathbb{R}^n$ be open and convex. If f is convex or concave, then it is continuous.*

Proposition 4.6. *Let $X \subset \mathbb{R}^n$ be open and convex. If f is twice continuously differentiable, then*

(i) f is concave if and only if $H_f(x)$ is negative semidefinite, for all $x \in X$.

(ii) f is convex if and only if $H_f(x)$ is positive semidefinite, for all $x \in X$.

(iii) if, for all $x \in X$, $H_f(x)$ is negative definite, then f is strictly concave.

(iv) if, for all $x \in X$, $H_f(x)$ is positive definite, then f is strictly convex.

Intuitively, just like $\nabla f(x)$ is the best linear approximation to f at x, $H_f(x)$ is the best quadratic approximation to f at x. Hence, if $H_f(x)$ is positive definite, so the quadratic form is convex, we conclude that f is *locally* convex at x. Thus, if f is *locally* convex at every point, it is *globally* convex. An analogous simile can be stated for the remaining characterizations.

Example 4.7. Let $\alpha_1, \alpha_2 > 0$. The function $f : \mathbb{R}^2_{++} \to \mathbb{R}$, $x = (x_1, x_2) \mapsto x_1^{\alpha_1} x_2^{\alpha_2}$ is concave if and only if $\alpha_1 + \alpha_2 \leq 1$. Indeed, The Hessian matrix of f is

$$H_f(x) = \begin{pmatrix} \frac{\alpha_1(\alpha_1 - 1) f(x)}{x_1^2} & \frac{\alpha_1 \alpha_2 f(x)}{x_1 x_2} \\ \frac{\alpha_1 \alpha_2 f(x)}{x_1 x_2} & \frac{\alpha_2(\alpha_2 - 1) f(x)}{x_2^2} \end{pmatrix},$$

for $x \in \mathbb{R}^2_{++}$. Now,

- $f_{11}(x), f_{22}(x) \leq 0$ if and only if $\alpha_1, \alpha_2 \leq 1$, and
- $|H_f(x)| = \alpha_1\alpha_2(1-\alpha_1-\alpha_2)\frac{f(x)}{(x_1x_2)^2} \geq 0$ if and only if $1-\alpha_1-\alpha_2 \geq 0$.

Thus, we conclude that f is concave if and only if $\alpha_1 + \alpha_2 \leq 1$.

Proposition 4.7. *Let $f, g : X \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ be functions. Then*

(i) If f and g are convex (concave), then $f + g$ is convex (concave).

(ii) If h is increasing and convex (concave) and f is convex (concave), then $h \circ f$ is convex (concave).

(iii) If h is decreasing and convex (concave) and f is concave (convex), then $h \circ f$ is convex (concave).

Proof. (i) Assume f and g are convex. Let $x_1, x_0 \in X$ and $t \in [0,1]$, then

$$\begin{aligned}(f+g)(x_t) &= f(x_t) + g(x_t)\\ &\leq t(f(x_1) + g(x_1)) + (1-t)(f(x_0) + g(x_0)), \text{ since } f, g \text{ are convex}\\ &= t(f+g)(x_1) + (1-t)(f+g)(x_0)\end{aligned}$$

In conclusion, $f + g$ is convex.

(ii) Assume h is increasing and convex, and f is convex. Let $x_1, x_0 \in X$ and $t \in [0,1]$. Since f is convex, then

$$f(x_t) \leq tf(x_1) + (1-t)f(x_0).$$

Now, since h is increasing

$$h(f(x_t)) \leq h(tf(x_1) + (1-t)f(x_0)),$$

and by convexity of h, we have

$$h(tf(x_1) + (1-t)f(x_0)) \leq th(f(x_1)) + (1-t)h(f(x_0)).$$

In conclusion, $(h \circ f)(x_t) \leq t(h \circ f)(x_1) + (1-t)(h \circ f)(x_0)$, which means that $h \circ f$ is convex.

(iii) Exercise.

□

4.3 Quasiconvex and quasiconcave functions

Definition 4.6. Let $X \subset \mathbb{R}^n$ be convex, and $f : X \to \mathbb{R}$ be a function.

(i) f is **quasiconvex** if $f(x_t) \leq \max\{f(x_1), f(x_0)\}$, for all $x_1, x_0 \in X$ and each $t \in [0, 1]$.

(ii) f is **strictly quasiconvex** if $f(x_t) < \max\{f(x_1), f(x_0)\}$, for all different $x_1, x_0 \in X$ and each $t \in (0, 1)$.

(iii) f is **quasiconcave** if $f(x_t) \geq \min\{f(x_1), f(x_0)\}$, for all $x_1, x_0 \in X$ and each $t \in [0, 1]$.

(iv) f is **strictly quasiconcave** if $f(x_t) > \min\{f(x_1), f(x_0)\}$, for all different $x_1, x_0 \in X$ and each $t \in (0, 1)$.

Proposition 4.8. *Every convex (concave) function is quasiconvex (quasiconcave).*

Proof. Assume f is convex. Let $x_1, x_0 \in X$ and $t \in [0, 1]$. Then,

$$\begin{aligned} f(x_t) &\leq tf(x_1) + (1 - t)f(x_0) \\ &\leq t \max\{f(x_1), f(x_0)\} + (1 - t) \max\{f(x_1), f(x_0)\} \\ &= \max\{f(x_1), f(x_0)\}. \end{aligned}$$

Hence, f is quasiconvex. $\square$

Proposition 4.9. *Every (strictly) monotone function $f : \mathbb{R} \to \mathbb{R}$ is both (strictly) quasiconvex and (strictly) quasiconcave.*

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be an increasing function, let $t \in [0, 1]$, and $x_1 \geq x_0$. Since $x_1 \geq x_t \geq x_0$, and f is increasing then $f(x_1) = \max\{f(x_1), f(x_0)\} \geq f(x_t) \geq f(x_0) = \min\{f(x_1), f(x_0)\}$. $\square$

Example 4.8. The function $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto e^x$ is convex, quasiconvex, and quasiconcave, but not concave.

Definition 4.7. Let $f : X \to \mathbb{R}$ be a function.

(i) The **upper contour set** of f at $\alpha \in \mathbb{R}$ is the set $U_f(\alpha) = \{x \in X \mid f(x) \geq \alpha\}$.

(ii) The **lower contour set** of f at $\alpha \in \mathbb{R}$ is the set $L_f(\alpha) = \{x \in X \mid f(x) \leq \alpha\}$.

Proposition 4.10. *Let $f; X \to \mathbb{R}$ be a function. Then f is quasiconvex (quasiconcave) if and only if $L_f(\alpha)$ ($U_f(\alpha)$) is convex, for all $\alpha \in \mathbb{R}$.*

Proof. Assume that f is quasiconvex. Fix $\alpha \in \mathbb{R}$. Let $x_1, x_0 \in L_f(\alpha)$ and $t \in [0, 1]$. Since f is quasiconvex, $f(x_t) \leq \max\{f(x_1), f(x_0)\} \leq \alpha$, so $f(x_t) \in L_f(\alpha)$.

Now, suppose that $L_f(\alpha)$ is convex for any $\alpha \in \mathbb{R}$. Let $x_0, x_1 \in X$ and $t \in [0, 1]$. Write $\alpha^* = \max\{f(x_1), f(x_0)\}$, so that $x_0, x_1 \in L_f(\alpha^*)$. Since $L_f(\alpha^*)$ is convex, then $x_t \in L_f(\alpha^*)$. In other words, $f(x_t) \leq \max\{f(x_1), f(x_0)\}$, which means that f is quasiconcave. $\square$

Example 4.9. Let $a_1, \ldots, a_n > 0$. The function $f : \mathbb{R}^n \to \mathbb{R}$, $x = (x_i)_{i=1}^n \to \min_{i=1,\ldots,n}\{a_i x_i\}$ is quasiconcave.

Fix $\alpha \in \mathbb{R}$. Observe that $x \in U_f(\alpha)$ means that $\min_{i=1,\ldots,n}\{a_i x_i\} \geq \alpha$. This is equivalent to $a_i x_i \geq \alpha$, for each $i = 1, \ldots, n$. In other words, $x_i \in \left[\frac{\alpha}{a_i}, \infty\right)$ for each $i = 1, \ldots, n$.

In conclusion, $U_f(\alpha) = \prod_{i=1}^n \left[\frac{\alpha}{a_i}, \infty\right)$, which is a convex set for being the Cartesian product of convex sets. Thus, Proposition 4.10 implies that f is quasiconcave.

Proposition 4.11. *Let $f : X \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ be functions. Then*

(i) If f is quasiconcave (quasiconvex) and h is increasing, then $h \circ f$ is quasiconcave (quasiconvex).

(ii) If f is quasiconcave (quasiconvex) and h is decreasing, then $h \circ f$ is quasiconvex (quasiconcave).

Proof. Exercise. □

Example 4.10. Let $\alpha, \beta > 0$. The function $f : \mathbb{R}^2_{++} \to \mathbb{R}$, $x = (x_1, x_2) \mapsto x_1^\alpha x_2^\beta$ is quasiconcave.

By Example 4.7, we know that the function $g : \mathbb{R}^2_{++}$, $x = (x_1, x_2) \mapsto x_1^{\frac{\alpha}{\alpha+\beta}} x_2^{\frac{\beta}{\alpha+\beta}}$ is concave, and therefore it is quasiconcave. On the other hand, the function $h : \mathbb{R} \to \mathbb{R}$, $x \mapsto x^{\alpha+\beta}$ is increasing, so that Proposition 4.11 ensures that $f = h \circ g$ is quasiconcave.

Example 4.11. The sum of quasiconvex (quasiconcave) functions is not necessarily quasiconvex (quasiconcave). To illustrate, consider $f(x) = x^3$ and $g(x) = -x$.

Definition 4.8. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function and $x \in \mathbb{R}^n$. The **bordered Hessian** of f at x is

$$\mathbb{B}_f(x) = \begin{pmatrix} 0 & f_1(x) & f_2(x) & \ldots & f_n(x) \\ f_1(x) & f_{11}(x) & f_{12}(x) & \ldots & f_{1n}(x) \\ f_2(x) & f_{21}(x) & f_{22}(x) & \ldots & f_{2n}(x) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_n(x) & f_{n1}(x) & f_{n2}(x) & \ldots & f_{nn}(x) \end{pmatrix} = \begin{pmatrix} 0 & \nabla f(x)' \\ \nabla f(x) & H_f(x) \end{pmatrix}_{(n+1)\times(n+1)}$$

Theorem 4.1. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be twice continuously differentiable. Then for $\mathbb{B}_f(x)$,*

(i) If $M_r(x) < 0$ for all $x \in \mathbb{R}^n$ and each $r = 2, \ldots, n+1$, then f is quasiconvex.

(ii) If $(-1)^r M_r(x) < 0$ for all $x \in \mathbb{R}^n$ and each $r = 2, \ldots, n+1$, then f is quasiconcave.

4.4 Exercises

(i) Let $x_0, x_1 \in \mathbb{R}$ be such that $x_0 < x_1$. Prove that $f : [0,1] \to [x_0, x_1]$ defined by $f(t) = x_t$ is continuous.

(ii) Complete the proof of Proposition 4.1.

(iii) Let $X \subset \mathbb{R}^n$. Prove that $Conv(X)$ is the smallest convex set containing X, i.e., for any convex set $C \subset R^n$, $X \subset C$ implies $Conv(X) \subset C$.

(iv) Complete the proof of Proposition 4.2.

(v) Let $x_0, x_1 \in \mathbb{R}^n$, $\epsilon > 0$, and $t \in [0,1]$. Prove that $B(x_t, \epsilon) = tB(x_1, \epsilon) + (1-t)B(x_0, \epsilon)$.

(vi) Let $X, Y \subset \mathbb{R}^n$. Prove that

  (a) If X is convex, and $X_1, X_0 \subset X$, then $X_t = tX_1 + (1-t)X_0 \subset X$.

  (b) If X is convex, then $\operatorname{int} X$ is convex. Is the converse true?

(vii) Let $X, Y \subset \mathbb{R}^n$. Prove that

  (a) Let X be convex, $t \in [0,1]$, and $X_0, X_1 \subset \mathbb{R}^n$. If $X \cap X_0 \neq \emptyset$ and $X \cap X_1 \neq \emptyset$, then $X_t \cap X \neq \emptyset$ where $X_t = tX_1 + (1-t)X_0$.

  (b) If X is convex, then $\overline{X}$ is convex. Is the converse true?

(viii) Prove Proposition 4.4.

(ix) Let $f, g : X \to \mathbb{R}$ be functions. Use Proposition 4.4 to prove that:

  (a) If f, g are convex, then $h = \max\{f, g\}$ is convex.

  (b) If f, g are concave, then $h = \min\{f, g\}$ is concave.

(x) Complete the proof of Proposition 4.7.

(xi) Let $f, g : X \to \mathbb{R}$ be functions. Use Proposition 4.10 to prove that:

  (a) If f, g are quasiconvex, then $h = \max\{f, g\}$ is quasiconvex.

  (b) If f, g are quasiconcave, then $h = \min\{f, g\}$ is quasiconcave.

(xii) Prove Proposition 4.11.

(xiii) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function. Say $x_0 \in \mathbb{R}^n$ is a **peak** if there exists $\epsilon > 0$ such that $f(x_0) > f(x_1)$ for all $x_1 \in B(x_0, \epsilon) \setminus \{x_0\}$.

Show that if f is quasiconcave then it is *single-peaked*; i.e., it does not admit more than one peak.

5 Optimization

Let $C \subset S \subset \mathbb{R}^n$, and $f : S \to \mathbb{R}^n$ be a function. In this section we focus on optimization problems. In particular, we are interested in maximize or minimize the values of the function f over the set C. We call f the **objective function**, and the set C the **constraint set**. When we attempt to maximize f over the set C we write

$$\max_{x \in C} f(x).$$

The minimization problem is posed in a similar fashion.

A **solution** to the maximization problem is a point $x \in C$ such that $f(x) \geq f(y)$ for all $y \in C$. We say that f attains a **maximum** on C at x, or that x is a **maximizer** of f on C. Write $\arg\max_{x \in C} f(x)$ for the set of all solutions to the maximization problem.

Similarly, a solution to the minimization problem $\min_{x \in C} f(x)$ is a point $x \in C$ such that $f(x) \leq f(y)$ for all $y \in C$. We say that f attains a **minimum** at x on C, or that x is a **minimizer** of f on C. Write $\arg\min_{x \in C} f(x)$ for the set of all solutions to the minimization problem.

Example 5.1. Let $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto x^2$. Then $\arg\max_{x \in [-1,1]} f(x) = \{-1, 1\}$.

Example 5.2. Let $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto x$. Since $\sup f(\mathbb{R}) = \infty$, then the maximization problem $\max_{x \in \mathbb{R}} f(x)$ has no solution.

Proposition 5.1. *$x \in C$ is a minimizer of f on C if and only if $x \in C$ is a maximizer of $-f$ on C.*

Proof. $x \in C$ is a minimizer of f on C means that $f(x) \leq f(y)$ for all $y \in C$. This is equivalent to $-f(x) \geq -f(y)$ for all $y \in C$, which is to say that $x \in C$ is a maximizer of $-f$ on C. □

Proposition 5.1 shows that any minimization problem can be posed in terms of maximization, and so that we can focus on maximization techniques without loss of generality.

Proposition 5.2. *Let $g : \mathbb{R} \to \mathbb{R}$ be a strictly increasing function. Then $x \in C$ is a maximizer of f on C if and only if $x \in C$ is a maximizer of $g \circ f$ on C.*

Proof. Exercise. □

5.1 Properties of the solution

Recall that the extreme value theorem (Corollary 2.4) ensures that $\arg\max_{x \in C} f(x)$ is nonempty provided that f is continuous and C is compact. Further,

Proposition 5.3. *(Extreme value theorem - revisited) If $f : S \to \mathbb{R}$ is continuous and C is compact, then $\arg\max_{x \in C} f(x)$ is a non-empty compact set.*

Proof. We know that $\arg\max_{x \in C} f(x) \subset C$. Since C is compact, it suffices that $\arg\max_{x \in C} f(x)$ is closed for it to be compact, in view of Proposition 2.19.

Let $(x_n)_{n \geq 1}$ be a sequence in $\arg\max_{x \in C} f(x)$ with $x_n \to x$. We need to prove that (i) $x \in C$, and (ii) $f(x) \geq f(y)$ for all $y \in C$. The former holds since $(x_n)_{n \geq 1}$ is a convergent sequence in C, and C is a closed set so it contains $\lim x_n$. The latter follows from the continuity of f and the definition of $\arg\max_{x \in C} f(x)$. As a matter of fact, continuity ensures that $f(x_n) \to f(x)$, and since $f(x_n)$ must be a constant for each $n \in \mathbb{N}$, then $f(x) = f(x_n) \geq f(y)$ for all $y \in C$. This completes the proof. □

Proposition 5.4. *(Uniqueness of the solution)*

Let $C, S \subset \mathbb{R}^n$ be convex sets with $C \subset S$, and $f : S \to \mathbb{R}$ be a function. If f is strictly quasiconcave, then $\arg\max_{x \in C} f(x)$ has at most one element.

Proof. By contradiction. Let $x_1, x_0 \in C$ be two different solutions to the maximization problem. Since f is strictly quasiconcave, then $f(x_{0.5}) > \min\{f(x_0), f(x_1)\} \geq f(y)$ for all $y \in C$. Since C is convex, then $x_{0.5} \in C$, contradicting that x_1, x_0 are solutions to the maximization problem. □

Proposition 5.5. *(Convexity of the solutions)*

Let $C, S \subset \mathbb{R}^n$ be convex sets with $C \subset S$, and $f : S \to \mathbb{R}$ be a function. If f is quasiconcave, then $\arg\max_{x \in C} f(x)$ is a convex set.

Proof. Exercise. □

Remark 5.1. Notice that Propositions 5.3, 5.4, and 5.5 provide sufficient conditions for existence, uniqueness, and convexity respectively but are silent on what happens when the conditions fail. In particular, if the conditions fail a maximizer may still exist, the maximizer may still be unique, and the solution set may still be convex.

5.2 Unconstrained optimization

This section concerns the set of maximizers in the interior of a feasible set. Fix $C \subset S \subset \mathbb{R}^n$, and let f be a real-valued function defined on S. Call the point $x^* \in C$ a **local maximizer** of f on C if there exists $\epsilon > 0$ such that $f(x^*) \geq f(x)$ whenever $x \in C$ and $d(x, x^*) < \epsilon$. If x^* is a local maximizer call the value $f(x^*)$ a local maximum. Likewise, define a **local minimizer** and a **local minimum**.

We usually say that a maximizer is a global maximizer to emphasize that it is not only a local maximizer. While we are interested in global maximizers, we will see that local maximizers are important objects in solving global optimization problems.

Proposition 5.6. *Every global maximizer is a local maximizer.*

Proof. By definition. $\square$

Proposition 5.7. *Let $C, S \subset \mathbb{R}^n$ be convex sets and $f : S \to \mathbb{R}$ be a function. If f is strictly quasiconcave and $x_0 \in C$ is a local maximizer of f on C, then it is a global maximizer of f on C.*

Proof. Exercise. $\square$

5.2.1 First-order conditions

Let $S \subset \mathbb{R}^n$ be open and $f : S \to \mathbb{R}$ a differentiable function. A point $x \in S$ is a **critical point** of f if its **gradient** is zero, i.e.

$$\nabla f(x) := \left(\tfrac{\partial f}{\partial x_1}(x), ..., \tfrac{\partial f}{\partial x_n}(x)\right) = (0, ..., 0).$$

Note that in the particular case $n = 1$, $\nabla f(x) = f'(x)$.

Proposition 5.8. *Let $I \subset \mathbb{R}$ be an interval, and let f be a real-valued function defined on I. If $x^* \in \text{int}I$ is a local maximizer of f on I and f is differentiable, then x^* is a critical point of f.*

Proof. Suppose that x^* is a local maximizer of f. Let $h > 0$ be sufficiently small so that $f(x^*) \geq f(x^* + h)$. and hence

$$\frac{f(x^*) - f(x^* + h)}{h} \geq 0.$$

Since h is arbitrarily small and f is differentiable, it follows that $f'(x^*) \geq 0$. By taking $h < 0$ sufficiently small, we can also conclude that $f'(x^*) \leq 0$. Hence, $f'(x^*) = 0$. $\square$

Example 5.3. The first-order conditions are necessary but not sufficient. To illustrate, consider $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto x^3$. So, $f'(x) = 3x^2$. Then, $x = 0$ satisfies the first-order condition $f'(x) = 0$ but is not a local maximizer.

Proposition 5.9. *Fix a subset $S \subset \mathbb{R}^n$ and let f be a real-valued function defined on S. If $x^* \in \text{int}S$ is a local maximizer of f and f is differentiable, then x^* is a critical point of f.*

Proof. The proof follows by applying Proposition 5.8 successively to the functions $x_j \mapsto f(x^* + he_j)$ and $x_j \mapsto f(x^* - he_j)$ for each $j = 1, ..., n$, where e_j is the j-th standard basis vector and $h \in \mathbb{R}$ is sufficiently small. $\square$

5.2.2 Sufficiency of first-order conditions

Proposition 5.10. *Fix a convex set $S \subset \mathbb{R}^n$, $f : S \to \mathbb{R}$ a differentiable function and $x^* \in \text{int}S$.*

(i) If f is concave or strictly quasiconcave, then x^* is a global maximizer of f in S if and only if x^* is a critical point.

(ii) If f is convex or strictly quasiconvex, then x^* is a global minimizer of f in S if and only if x^* is a critical point.

Remark 5.2. Recall that Proposition 4.6 states that a function f is convex in S if and only if $H_f(x)$ is positive semidefinite, for all $x \in S$. Likewise, f is concave in S if and only if $H_f(x)$ is negative semidefinite, for all $x \in S$.

5.2.3 Second-order conditions

Proposition 5.10 states that the first-order conditions are sufficient to identify interior optima in functions that are globally concave or convex. The following results relax such an assumption and identify all interior local optima.

Proposition 5.11. *Fix a convex set $S \subset \mathbb{R}^n$, $f : S \to \mathbb{R}$ a continuously differentiable function and $x \in \text{int}S$.*

(i) If x^* is a local maximizer of f in S, then $H_f(x)$ is negative semidefinite.

(ii) If x^* is a local minimizer of f in S, then $H_f(x)$ is positive semidefinite.

Proposition 5.11 does not tell us which points are local optima, but only which ones are not. The following Proposition concerns sufficiency.

Proposition 5.12. *Let $S \subset \mathbb{R}^n$ be open, $f : S \to \mathbb{R}$ be a twice continuously differentiable function, and $x \in \text{int}\, S$.*

(i) If $\nabla f(x) = 0$ and $H_f(x)$ is negative definite, then x is a local maximizer.

(ii) If $\nabla f(x) = 0$ and $H_f(x)$ is positive definite, then x is a local minimizer.

Notice however that this last proposition is silent about identifying optima on the boundary of S. The following subsection's goal is to identify optima that are on the boundary of a given set by parametrizing it in terms of a set of functions.

5.3 Constrained Optimization

Let S be an open subset of $\mathbb{R}^n$. Let $f : S \to \mathbb{R}$ and $g_j : S \to \mathbb{R}$, $j = 1, ...m$ be continuously differentiable functions. Consider an optimization problem of the form

$$\begin{aligned} \max_{x \in S} \quad & f(x) \\ \text{subject to} \quad & g_j(x) = 0 \text{ for } j = 1, ..., m \end{aligned}$$

with $m \leq n$.

5.3.1 An Example in $\mathbb{R}^2$

To grasp some intuition we will provide a simple example in $\mathbb{R}^2$ with $m = 1$ and we will generalize after that. Consider the problem of the form

$$\begin{aligned} \max_{x \in S} \quad & f(x) \\ \text{subject to} \quad & g(x) = 0 \end{aligned}$$

for simplicity assume that $g(x) = x_1^2 + x_2^2 - 1$ so the constraint $g(x) = 0$ represents the unitary circle. Assume also that f is strictly increasing in both arguments.

Remark 5.3. (i) *Direction of steepest ascent.* Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function and $a \in \mathbb{R}^n$. Recall that any partial derivative can be interpreted as the derivative of an auxiliary univariate function. Hence, $\frac{\partial f}{\partial x_i}(a)$ represents the slope of the tangent line to f at a in the direction of the x_i axis. However, we could ask for the slope of the tangent line to f at a in any direction we wish, not only focusing on the axes. Let $u \in \mathbb{R}^n$ be an unit vector (i.e., $\|u\| = 1$), we define the **directional derivative** of f at a in the direction of u as the limit

$$D_u f(a) = \lim_{h \to 0} \frac{f(a + hu) - f(a)}{h}$$

Observe that $D_{e_i} f = f_{x_i}$. When f is differentiable, it holds that $D_u f(a) = \nabla f(a) \cdot u$ for all unit vectors $u \in \mathbb{R}^n$.

Now, recall that the dot product between two vectors $x, y \in \mathbb{R}^n$ satisfies $x \cdot y = \cos(\theta)\|x\|\|y\|$, where θ is the angle formed by them. Hence, for any unit vector $u \in \mathbb{R}^n$, $D_u f(a) = \nabla f(a) \cdot u = \cos(\theta)\|\nabla f(a)\|$. Thus, the direction in which the function would increase the most is when $\theta = 0$ (so $\cos(\theta) = 1$), that is when u is in the direction of the gradient $\nabla f(a)$. Put differently, the gradient $\nabla f(a)$ points in the direction of the steepest ascent.

(ii) *The gradient is perpendicular to level surfaces.* Let $g : \mathbb{R}^n \to \mathbb{R}$ be a function, and consider the level surface given by $g(x) = c$. Let $x : \mathbb{R} \to \mathbb{R}^n$ be a parameterized curve on the level surface $g(x) = c$. Since $x(t)$ is on the level surface then $g(x(t)) = c$. Differentiating both sides with respect to t we obtain $\nabla g(x(t)) \cdot x'(t) = 0$ for any t. As $x'(t)$ represents the slope of the tangent line to the curve $x(t)$, then $\nabla g(x(t))$ is an orthogonal vector to the curve $x(t)$. Finally, since $x(t)$ was an arbitrary curve on the level surface $g(x) = c$, it follows that $\nabla g(x)$ is an orthogonal vector to the surface level $g(x) = c$ at any point x.

Recall that the gradient of f is the direction of the local maximal growth of the function. In addition, the gradient of g illustrates a direction that is perpendicular to the curve $g(x) = 0$. So, the key idea to find a constrained optimum x^* is that the gradients ∇g and ∇f must be multiples of each other. (See the left panel of Figure 5.1.) On the other hand, if the gradients are not multiples of each other, there is *path* among the curve $g(x) = 0$ that increases the value of f. (See point x^* at the right panel of Figure 5.1.)

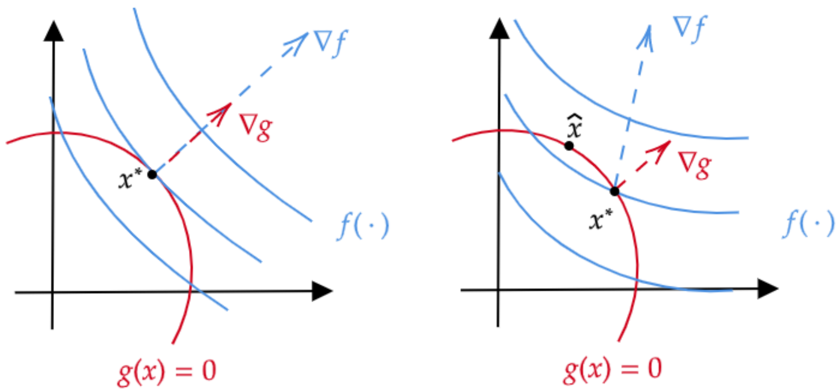


Figure 5.1.

We write this condition as $\nabla f(x^*) + \lambda \nabla g(x^*) = 0$. This condition, together with the requirement that $g(x^*) = 0$ can be written as follows:

$$\begin{aligned} \frac{\partial f}{\partial x_1}(x^*) + \lambda \frac{\partial g}{\partial x_1}(x^*) &= 0, \\ \frac{\partial f}{\partial x_2}(x^*) + \lambda \frac{\partial g}{\partial x_2}(x^*) &= 0, \\ g(x^*) &= 0 \end{aligned}$$

For instance in the case $f(x_1, x_2) = x_1 + x_2$, the vectors $(-1, -1)$ and $(1, 1)$ are the only two vectors that satisfy the conditions above. Notice that the set $\{(x_1, x_2) \in \mathbb{R}^2 : g(x_1, x_2) = 0\}$ is compact. Thus, since f is continuous, a maximum and a minimum exist. Since $f(-1, -1) < f(1, 1)$, it follows that $(1, 1)$ is the only solution to the optimization problem.

5.3.2 The Lagrangian in $\mathbb{R}^n$ with one constraint

We now generalize to general functions in n dimensions. First note that the condition $\nabla f(x^*) - \lambda \nabla g(x^*) = 0$ can be conveniently viewed as the first-order conditions of the objective function

$$\mathcal{L}(x, \lambda) = f(x) + \lambda g(x).$$

So, the gradient of $\mathcal{L}$ with respect to x at point x^* and multiplier λ^* is

$$\nabla_x \mathcal{L}(x^*, \lambda^*) = \nabla f(x^*) + \lambda \nabla g(x^*)$$

We call this objective function the **Lagrangian** and we call the number λ the **Lagrangian multiplier.** The interpretation of this objective function is the same term $f(x)$ that is desired to maximize plus a punishment of $\lambda g(x)$ for violating the constraint $g(x) = 0$. The main result of this section intuitively states that we can set penalties for violating the equality constraint in such a way that at the maximum the constraint is exactly satisfied. For this reason, given an optimum x^*, we call the associated Lagrange multiplier λ the **shadow value** of the constraint g. So, it represents the increase of the objective function generated by a marginal relaxation of the constraint g.

Proposition 5.13. *Let S be an open subset of $\mathbb{R}^2$, and let $f : S \to \mathbb{R}$ and $g : S \to \mathbb{R}$ be continuously differentiable functions. If $x^* = (x_1^*, x_2^*) \in S$ is a solution to the problem*

$$\begin{aligned} \max_{x \in S} \quad & f(x) \\ \textit{subject to} \quad & g(x) = 0 \end{aligned}$$

and $\nabla g(x^*) \neq 0$, then there exists a unique $\lambda^* \in \mathbb{R}$ such that x^* satisfies

$$\begin{aligned} \nabla_x \mathcal{F}(x^*, \lambda^*) = \nabla f(x^*) + \lambda \nabla g(x^*) = 0 \\ g(x^*) = 0 \end{aligned}$$

For proof of this and the rest propositions in this section see Sundaram et al. (1996). Note that Proposition 5.13 states necessary conditions for a point to be a maximum, so it actually does not help to decide which points are maxima, and rather lets us learn which are not. The following theorem provides sufficient conditions for a point to be a local maximum.

Proposition 5.14. *Let $S \subset \mathbb{R}^2$ be open, $f, g : S \to \mathbb{R}$ be twice continuously differentiable functions, and (x^*, λ^*) points satisfying Proposition 5.13. If $|H_{\mathcal{L}}(x^*, \lambda^*)| > 0$, then x^* is a local maximum.*

Remark 5.4. In the case $|H_{\mathcal{L}}(x^*)| < 0$, x^* is a local minimum. If the determinant is zero, the criterion does not conclude.

Example 5.4. Let $u : \mathbb{R}^2 \to \mathbb{R}$ be a twice continuously differentiable function, and p_x, p_y, w positive constants. Consider the maximization problem

$$\begin{aligned} \max_{x,y>0} \quad & u(x, y) \\ \text{subject to } & p_x x + p_y y - w = 0 \end{aligned}$$

The associated Lagrangian function is $\mathcal{L}(x, y, \lambda) = u(x, y, \lambda) = u(x, y) - \lambda(p_x x + p_y y - w)$, so that

$$H_{\mathcal{L}} = \begin{pmatrix} u_{xx} & u_{xy} & p_x \\ u_{yx} & u_{yy} & p_y \\ p_x & p_y & 0 \end{pmatrix}$$

Hence, $|H_{\mathcal{L}}| = p_x p_y(u_{yx} + u_{xy}) - p_x^2 u_{yy} - p_y^2 u_{xx}$, which is the quadratic form

$$\begin{pmatrix} p_y & p_x \end{pmatrix} \begin{pmatrix} -u_{xx} & u_{xy} \\ u_{yx} & -u_{yy} \end{pmatrix} \begin{pmatrix} p_y \\ p_x \end{pmatrix}$$

Thus, for a point (x^*, y^*) satisfying Proposition 5.13 to be a solution to the problem, it suffices that $|H_{\mathcal{L}}(x^*, y^*)| > 0$, which is to say that the quadratic form is positive definite:

(i) $-u_{xx}(x^*, y^*) > 0$, and

(ii) $u_{xx}(x^*, y^*)u_{yy}(x^*, y^*) - u_{xy}(x^*, y^*)u_{yx}(x^*, y^*) > 0$.

Observe that the above conditions are equivalent to $H_u(x^*, y^*)$ being negative definite. Therefore, it suffices that u is locally strictly concave at (x^*, y^*) for it to be a solution to the problem.

5.3.3 The General Case

Now we proceed to analyze the general case with multiple restrictions given by $g_i(x) = 0$ for $i = 1, ..., m$. The Lagrangian with n restrictions is defined as

$$\mathcal{L}(x, \lambda) = f(x) + \sum_{j=1}^{m} \lambda_i g_i(x).$$

So, in this case, there is a Lagrangian multiplier for each constraint, representing the punishment/reward of violating/satisfying each constraint.

Definition 5.1. Fix m differentiable functions $g_i : \mathbb{R}^n \to \mathbb{R}, i = 1, ..., m$. The Jacobian matrix of $(g_1, ..., g_m)$ at the point $x \in \mathbb{R}^n$ is the matrix $J(x)$ whose i-th row is the gradient of g_i at x. So, in matrix notation,

$$J(x) = \begin{bmatrix} \frac{\partial g_1}{\partial x_1}(x) & \cdots & \frac{\partial g_1}{\partial x_n}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial g_m}{\partial x_1}(x) & \cdots & \frac{\partial g_m}{\partial x_n}(x) \end{bmatrix}$$

Proposition 5.15. *Let S be an open subset of $\mathbb{R}^n$, and let $f : S \to \mathbb{R}$ and $g_i : S \to \mathbb{R}$, $i = 1, ..., m$ be continuously differentiable functions with $m \leq n$. If $x^* \in S$ is a solution to the problem*

$$\begin{aligned} \max_{x \in S} \quad & f(x) \\ \textit{subject to} \quad & g_i(x) = 0 \textit{ for } i = 1, ..., m \end{aligned}$$

and the Jacobian matrix of $(g_1, ..., g_m)$ at the point x^* has rank m.[5] *Then, there exists a unique $\lambda^* = (\lambda_1^*, ..., \lambda_m^*)$ such that x^* satisfies*

$$\nabla_x \mathcal{L}(x^*, \lambda^*) := \nabla f(x^*) + \sum_{i=1}^{m} \lambda_i \nabla g_i(x^*) = 0$$

$$g_i(x^*) = 0 \qquad \textit{for each } i = 1, ...m$$

Example 5.5. This example shows the necessity of the rank condition of the Jacobian matrix. Let $f(x_1, x_2) = x_2$, $g_1(x_1, x_2) = x_1$ and $g_2(x_1, x_2) = -x_1 - x_2$. Consider the problem

$$\begin{aligned} \max_{(x_1,x_2)\in\mathbb{R}^2} \quad & f(x_1, x_2) \\ \text{subject to} \quad & g_1(x_1, x_2) = 0 \\ & g_2(x_1, x_2) = 0. \end{aligned}$$

Notice that the only point in $\mathbb{R}^2$ that satisfies the constraints is $(0, 0)$; so $(0, 0)$ must be the unique solution to the problem. Notice also that

$$\nabla f(x_1, x_2) = (0, 1) \qquad \nabla g_1^*(x_1, x_2) = (1, 0) \qquad \nabla g_2^*(x_1, x_2) = (-1, -2x_2)$$

So, the Jacobian matrix at (0,0) is

$$J(0, 0) = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$$

which rank is $1 < 2$. So, note that there are no (λ_1, λ_2) such that

$$\nabla f(0, 0) + \lambda_1 \nabla g_1(0, 0) + \lambda_2 \nabla g_2(0, 0) = 0.$$

So, although $(0, 0)$ solves the problem, it does not satisfy the first-order condition of the Lagrangian. Thus, in order to find the true optima of one specific problem, the analyst needs to find all points x that satisfy the Lagrangian FOC and in addition, verify all the points where the Jacobian does not have full rank.

5.3.4 Inequality constraints

Let S be an open subset of $\mathbb{R}^n$ and let f and g_i, $i = 1, ...m$ be real-valued continuously differentiable functions defined on S. In this section, we turn to optimization problems of the

[5]The **rank** of a matrix is the largest number of columns that constitute an l.i. set.

form

$$\max_{x \in S} \quad f(x),$$
$$\text{subject to} \quad g_i(x) \geq 0 \text{ for } i = 1, ...m.$$

To grasp some intuition, first, consider the problem with a unique constraint $g(x) \geq 0$. Say that the constraint defined by g is **binding** at x if $g(x) = 0$ and is **slack** (or **nonbinding**) if $g(x) > 0$.

Fix a solution $x^* \in \mathbb{R}^n$ to the constrained problem. Notice that if one constraint is slack at x^*, then the constraint $g(x) > 0$ has "room to spare" in the sense that if $g(x^*)$ decreases by a small amount the constraint is still satisfied. By contrast, if the constraint is binding at $x^* \in \mathbb{R}^n$, then there is "no room" to relax the value of g by any amount.

Thus, if the constraint is slack at x^*, then x^* is an interior solution with respect to g. So in that case we know that the first-order condition is simply $\nabla f(x^*) = 0$. Therefore, the Lagrangian multiplier λ does not appear in the condition and can take any value. Since $\lambda g(x)$ is interpreted as the punishment for violating the constraint g it is convenient to set $\lambda = 0$.

Now, if the constraint is binding at x^*, then $g(x^*) = 0$, which is the same condition as before. So, under rank condition of the Jacobian,

$$\nabla_x \mathcal{L}(x^*, \lambda^*) = 0$$

Moreover, in this case it must be $\lambda^* \geq 0$. Otherwise, the gradients of f and g point roughly to the same direction so there is a path where one can increase the value of $f(x)$ while satisfying the constraint $g(x) \geq 0$.

Thus, the conditions

$$\begin{aligned} \nabla_x \mathcal{L}(x^*, \lambda^*) &= 0 \\ \lambda^* &\geq 0 \\ g(x^*) &\geq 0 \\ \lambda^* g(x^*) &= 0 \end{aligned}$$

cover both cases. Additionally, when the constraint is binding we have $g(x^*) = 0$ and when the constraint is slack we have $\lambda^* = 0$. The constraint $\lambda^* g(x^*) = 0$ is called the **complementary slackness condition**. Note that this condition does not rule out he possibility that $\lambda^* = g(x^*) = 0$.

Before we state the result, we introduce some regularity conditions, which are usually called the **constraint qualifications (CQ)**, which guarantee necessity of the first-order conditions. The constraint qualifications are the following:

(A) Let k denote the number of constraints binding at x^*, then $k \leq n$ and the Jacobian matrix of $(g_1, ..., g_k)$[6] at the point x^* has k linearly independent columns.

(B) S is convex, and g_i is convex for $i = 1, ..., m$

(C) S is convex, and g_i is concave for $i = 1, ..., m$ and the staler's condition holds: there exists $\hat{x} \in S$ such that $g_i(\hat{x}) > 0$ for all $i = 1, ...m$.

Proposition 5.16. *Let S be an open subset of $\mathbb{R}^n$, and let $f : S \to \mathbb{R}$ and $g_i : S \to \mathbb{R}$, $i = 1, ..., m$ be continuously differentiable functions with $m \leq n$. If $x^* \in S$ and it is a solution to the problem*

$$\begin{aligned} \max_{x \in S} \quad & f(x) \\ \textit{subject to} \quad & g_i(x) \geq 0 \textit{ for } i = 1, ..., m, \end{aligned}$$

and one of the constraint qualifications is satisfied. Then, there exists a unique $\lambda^* = (\lambda_1^*, ..., \lambda_m^*)$ all greater than or equal to zero such that x^* satisfies

$$\begin{aligned} \nabla_x \mathcal{L}(x^*, \lambda^*) &= 0 \\ g_i(x^*) &\geq 0 \quad \textit{for each } i = 1, ...m \\ \lambda_i g_i(x^*) &= 0 \quad \textit{for each } i = 1, ...m. \end{aligned}$$

Notice that Proposition 5.16 can be generalized to a mixed case of some equality constraints and some inequality constraints. The easiest way to do it is to take all equalities to inequalities, and require that $\lambda^* > 0$ for any solution x^* you may find.

5.4 The Envelope Theorem

In many applications, the analyst is interested in the behavior of how the solution of a given optimization problem changes with respect to a certain parameter. Fix $S \subset \mathbb{R}^n$ and $\Theta \subset \mathbb{R}$ open sets and $f : S \times \Theta \to \mathbb{R}$. We are interested in analyzing the behavior of the problem given by

$$\max_{x \in S} f(x, \theta) \tag{1}$$

as θ varies in Θ.

Fix $S \subset \mathbb{R}^n$ and $\Theta \subset \mathbb{R}$ be open sets, and $f : S \times \Theta \to \mathbb{R}$ be a function. The **envelope function** of f is the function $F : S \times \Theta \to \mathbb{R}$ given by

$$F(\theta) = \sup_{x \in S} f(x, \theta).$$

[6]without loss of generality we let the first $k \leq m$ constraints to be the binding ones.

Say $\hat{x} : \Theta \to \mathbb{R}^n$ is a **solution** to the problem (1) if $f(\hat{x}(\theta)) = F(\theta)$ for each $\theta \in \Theta$.

Proposition 5.17. *(Envelope theorem)*

Let $S \subset \mathbb{R}^n$ and $\Theta \subset \mathbb{R}$ be open sets, and $f : S \times \Theta \to \mathbb{R}$ be a continuously differentiable function. If the solution function $\hat{x} : \Theta \to \mathbb{R}$ is differentiable at $\theta^* \in \Theta$, then $F'(\theta^*) = \frac{\partial f}{\partial \theta}(x^*(\theta^*), \theta^*)$.

Proof. Exercise. □

5.5 Exercises

(i) Prove Proposition 5.2. Does the result hold when g is weakly increasing?

(ii) Prove Proposition 5.5.

(iii) Prove Proposition 5.7.

(iv) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a quasiconcave function. Is it true that if $x_1 \in \mathbb{R}^n$ is a local maximizer, then it is a global maximizer?

(v) Fix $y \in \mathbb{R}^n$ and let $f(x) = -\sum_{i=1}^n (x_i - y_i)^2$. Find the global maximum of f in $\mathbb{R}^n$. Does f have any local minimum?

(vi) Let $f(x, y) = x^\alpha y^\beta$ and let $g(x, y) = px + qy - w$ with $\alpha, \beta, p, q, w \geq 0$. Solve the following problem:

$$\max_{(x,y)\in\mathbb{R}^2_+} f(x, y) \qquad \text{subject to } g(x, y) = 0$$

(vii) Let $f(x, y) = px + qy$ and let $g(x, y) = x^\alpha y^\beta = z$ with $\alpha, \beta, p, q, w \geq 0$. Solve the following problem:

$$\min_{(x,y)\in\mathbb{R}^2_+} f(x, y) \qquad \text{subject to } g(x, y) = 0$$

(viii) Write $f(x, y) = x^3 + y^3$ and $g(x, y) = x - y$. Consider the problem of maximizing and minimizing f over g. Show that $(0, 0)$ satisfy the first-order conditions. Does a minimum exist? Does a maximum exist? Why?

(ix) Prove Proposition 5.17.

(x) Consider the problem $\max_{\mathbb{R}} f(x, \theta)$ with $f(x) = f(x, \theta) = x^{\frac{1}{2}} - \theta x$ use the envelope theorem to find the derivative of the envelope function F of f.

References

Rangarajan K Sundaram et al. *A first course in optimization theory.* Cambridge university press, 1996.